Today: Elementary row operations ✓
Gaussian elimination ✓
(Reduced) row echelon form of a matrix. ✓
solving systems of equations ✓
Rank. ✓

Recall from last class

Ex: $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ — system of equations from last time.

$Ax = b$

we did: make an augmented matrix

$$(A \mid b) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 3 & 2 & 1 & 0 \end{array}\right)$$

b

$$\underset{\text{elem. row operations}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & & 0 & 1 \\ & 1 & & -3 \\ 0 & & 1 & -3 \end{array}\right) \rightsquigarrow \boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= -3 \\ x_3 &= -3 \end{aligned}}$$

solution. (unique!)

Note: in matrix form, we write it: (in this case, $A: \mathbb{R}^3 \to \mathbb{R}^3$ is an isomorphism):

if A^{-1} exists (!)

$$Ax = b$$

$$\underbrace{A^{-1} \cdot A}_{= \mathrm{Id}} \cdot x = A^{-1} b$$

$$\boxed{x = A^{-1} b}$$

(if we had A^{-1}, it would be easy to find the solution for every b)

How to invert a matrix using elementary row operations (5.4 in Jänisch)

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

our A — augmented: corresponds to solving three systems of equations at once:

$x = A^{-1}e_1 \longrightarrow Ax = e_1$

$x = A^{-1}e_2 \longrightarrow Ax = e_2$

$x = A^{-1}e_3 \longrightarrow Ax = e_3$

first column of A^{-1}

do row operations to get $\left(\begin{array}{ccc|c} 1 & 0 & & \\ & 1 & & \\ 0 & & 1 & \end{array}\right)$ whatever we get here is A^{-1}.

(exer.)

Why this works: row operations $\Leftrightarrow$ manipulating equations

get $\mathrm{Id} \cdot x = (\quad)$ $\uparrow$ new RHS.

$A^{-1}b$

clever observation: if we take $b_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$b_2 = e_2$ ⋮

$b_3 = e_3$ ⋮

This will find $A^{-1}e_1$ ← 1^{st} column of A^{-1}

$A^{-1}e_2$

$\vdots$

$$\left(A \;\middle|\; \begin{matrix}1\\0\\0\end{matrix} \right) \rightsquigarrow \left(Id \;\middle|\; \begin{matrix}*\\ *\\ *\end{matrix} \right)$$

solution to $Ax = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$= \text{exactly } A^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix}$

$= 1^{st}$ column of A^{-1}.

How can we get stuck?

Example $A = \begin{pmatrix}1 & -1 & 0\\ 2 & 0 & 2\\ 0 & 2 & 2\end{pmatrix}$

Try inverting it:

$$\left(\begin{matrix}1 & -1 & 0\\ 2 & 0 & 2\\ 0 & 2 & 2\end{matrix}\;\middle|\;\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right) \xrightarrow[R_2-2R_1]{} \left(\begin{matrix}1 & -1 & 0\\ 0 & 2 & 2\\ 0 & 2 & 2\end{matrix}\;\middle|\;\begin{matrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right)$$

already 1. (otherwise get 1 there)

need to make it 1

$$\xrightarrow[\frac{1}{2}R_2]{} \left(\begin{matrix}1 & -1 & 0\\ 0 & 1 & 1\\ 0 & 2 & 2\end{matrix}\;\middle|\;\begin{matrix}1 & 0 & 0\\ -1 & 1/2 & 0\\ 0 & 0 & 1\end{matrix}\right)$$

$$\xrightarrow[R_3-2R_2]{} \left(\underbrace{\begin{matrix}1 & -1 & 0\\ 0 & 1 & 1\\ 0 & 0 & \boxed{0}\end{matrix}}_{M}\;\middle|\;\begin{matrix}1 & 0 & 0\\ -1 & 1/2 & 0\\ 2 & -1 & 1\end{matrix}\right)$$

! ← matrix is NOT invertible

what does this mean:

1) some equations will have no solution:

$Ax = b$ ← depending on b, might have no solution:

e.g. $Mx = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ ← to have solutions, need 0 here *after* reductions

$\boxed{\text{our } A \text{ *after* reductions!}}$

2) if after reductions, got $Mx = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$

then we have: $x_1 - x_2 = c_1$
$x_2 + x_3 = c_2$

$M = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\xrightarrow{R_1+R_2}$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ← make it 0.

from last page

$\begin{cases} x_1 + x_3 = c_1 + c_2 \\ x_2 + x_3 = c_2 \end{cases}$

cannot get rid of these!

We got a 1-parameter family of solutions

$x_1 = c_1 + c_2 - x_3$ ← determines the "y" in the kernel of A.
$x_2 = c_2 - x_3$ ← "free variable" (can rename it "t" or smth.)

$x_0 = \begin{pmatrix} c_1 + c_2 \\ c_2 \end{pmatrix}$

$v = x_3 \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} \in \ker(A)$.

All this says, is:

$A : \mathbb{R}^3 \to \mathbb{R}^3$

we discovered that $\dim(\mathrm{Im}(A)) = 2$

$\dim(\ker(A)) = 1$

$Mx = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$

(2-dim image.)

get 1-parameter family of solutions when we have them

row operations change the image but <u>not</u> its dimension.

<u>The kernel</u>: Recall that any solution has the form $x = x_0 + y$, $y \in \ker A$.

$Ax = b$

x_0: some <u>one</u> solution: $Ax_0 = b$

Row operations help us find ~~the~~ a basis for $\ker(A)$: here it is $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

One more comment about $\mathrm{rank}(A) = \dim(\mathrm{Im}(A))$

- we could do col<u>umn</u> operations to find a basis for the image of A. (if the columns are lin. indep. they al<u>ready</u> give a basis for $\mathrm{Im}(A)$)
- After we did row operations to get A into "echelon form": $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

0s

$\mathrm{rk}(A) = \#$ number of nonzero rows.

Where it can go wrong:

after some operations, could get:

$$\begin{pmatrix} 1 & * & * & * & \\ 0 & 1 & * & & \\ 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & 0 & 1 \\ 0 & 0 & 0 & & \end{pmatrix}$$

← look for $\neq 0$ in this row in further column

(or permute rows below)

Pivot positions

all zeroes below the diagonal and **on** the diagonal.

leave it there, move to the next column.

column 4 has no pivot.

"Echelon form"

Will post a note.