

So far, we have

- $A: V \rightarrow W$ - linear map
choose a basis in V choose a basis in W \rightarrow get a matrix for A
(depends on the bases)

(if $V=W$, $A: V \rightarrow V$, then has to be the same basis)

- A matrix $(m \times n)$ gives a linear map from F^n to F^m .

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} r_1 \cdot x \\ \vdots \\ r_m \cdot x \end{pmatrix}$$

write vectors as columns

$$r_i \cdot x = a_{i1} x_1 + \dots + a_{in} x_n$$

- $A: V \rightarrow W$, $B: W \rightarrow U$ \rightarrow composition
 $\begin{matrix} n & m & r \end{matrix}$ $B \circ A: V \rightarrow U$
corresponds to the matrix product, BA .
? $r \times n$ -matrix.

Kernels and images

$$\text{rk}(A) = \dim(\text{Im}(A)) = \dim(\text{space spanned by columns of } A)$$

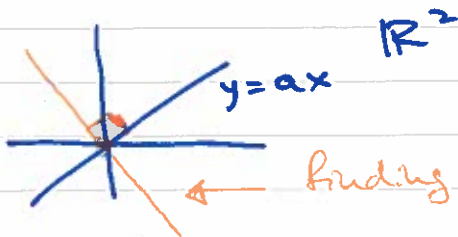
$$\dim(\text{Ker}(A)) + \text{rk}(A) = \dim(V)$$

In practice, we often care about ker(A).

In fact, if you have dot product,

- $\ker(A)$ = the space of all vectors in V perpendicular to all rows of A
(last class)

(This gives us geometric intuition in \mathbb{R}^n but not so useful for computation).



finding "orthogonal complement" is tricky.

We need: to solve equations. (7.1 in Jänisch)

We can encode systems of linear equations in this new language:

Example:
$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 + x_2 - x_3 = 2 \\ 3x_1 + 2x_2 + x_3 = 0 \end{cases}$$

We can write it as:
$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

the rows are coefficients of the equations. \leftarrow "A" \leftarrow "b"

$$\boxed{Ax = b} \leftarrow \text{our system of linear equations}$$

Possibilities for the solutions:

1. If $b = 0$, then
the solutions form a linear subspace:
the kernel of A .

2. Claim: If x_0 is one solution to $Ax = b$
then all solutions can be obtained as
 $x_0 + x$, where $x \in \text{Ker}(A)$.

Indeed, $A(x_0 + x) = Ax_0 + Ax = Ax_0 = b$.
② $\leftarrow x \in \text{Ker}(A)$

③ Suppose y is a solution:

$$Ay = b$$

we also have $Ax_0 = b$.

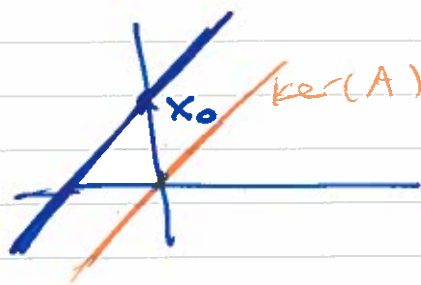
Then $Ay - Ax_0 = 0$

$$A(y - x_0) = 0 \Leftrightarrow y - x_0 \in \text{Ker}(A)$$

$$\Leftrightarrow y = x_0 + (y - x_0)$$

$x \in \text{Ker}(A)$.

* The solutions of any system of linear equations
form "a translated subspace" of V :



So, in \mathbb{R}^3 , it is always:

- a single point
- a line
- a plane
- (or \mathbb{R}^3 if $A = 0$)

Thus, we care about $\dim(\text{Ker}(A))$

For any vector b on the right, the dimension of the "translated space" of solutions is the same! ($= \dim(\ker(A))$)

"number of parameters you need to describe all solutions."

- Finding the solutions in parametric form \Leftrightarrow finding one solution x_0 , and a basis of $\ker(A)$.

(parametric form: $x_0 + t_1 v_1 + \dots + t_k v_k$ where $\{v_i\}$ form a basis of $\ker(A)$).

- If we only care about dimension, enough to find $\text{rk}(A)$ (which is easier) and use that $\dim(\ker(A)) = \dim(V) - \text{rk}(A)$

How to do this

- Elementary row (and column) operations.

3 operations:

1) Exchange two rows ~~XXXX~~ $R_i \leftrightarrow R_j$

2) multiply any row by $\neq 0$ scalar:
 $R_i \rightarrow \lambda R_i$

3) take R_i, R_j , and add λR_j to R_i
leave R_j unchanged.

Example $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} = A$

illustrate (3): $\underline{R_2} - 2\underline{R_1} \rightarrow$ this becomes R_1 is untouched!

Get $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 3 & 2 & 1 \end{pmatrix} \left| \begin{array}{l} 1 \\ 2 \\ 0 \end{array} \right| \begin{array}{l} 1 \\ 2 \\ 0 \end{array} = 0$
 $-1 - 2 \cdot 1 = -3$

$1 - 2 \cdot (-1) = 3$

At the level of equations, we have:

The second equation now doesn't involve x_1 .

This is elimination of variables!

Gaussian ~~matrix~~ elimination

- exchange rows if needed to have $a_{11} \neq 0$.
(maybe exchange columns if exchanging rows is not enough).
- multiply R_1 by $\frac{1}{a_{11}}$, to get the first entry to be 1.
- do operation (3) to rows to make the rest of the first column = 0. \uparrow only do $R_i - \lambda R_1$

In our example:

already 1 $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \left| \begin{array}{l} 1 \\ 2 \\ 0 \end{array} \right| \begin{array}{l} 1 \\ 2 \\ 0 \end{array} \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 3 & 2 & 1 \end{pmatrix} \left| \begin{array}{l} 1 \\ 0 \\ 0 \end{array} \right| \begin{array}{l} 1 \\ 0 \\ 0 \end{array} \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 5 & -2 \end{pmatrix} \left| \begin{array}{l} 1 \\ 0 \\ -3 \end{array} \right| \begin{array}{l} 1 \\ 0 \\ -3 \end{array}$

• Exchange things around to make $a_{22} \neq 0$.
if possible.

• Divide R_2 by a_{22} , to get 1 there.

• Now use row operations to kill the second column (except a_{22})

$$\leadsto \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 5 & -2 & -3 \end{array} \right) \xrightarrow{\substack{\text{leave } R_2 \\ R_1 + R_2 \\ R_3 - 5R_2}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -3 \end{array} \right)$$

intended luck
↓ ↓

$$\leadsto \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 \end{array} \right) \xrightarrow{R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \end{array} \right)$$

New system: $\begin{cases} x_1 = 1 \\ x_2 = -3 \\ x_3 = -3 \end{cases}$ - a solution!

Did 5.3, 7.1

Will do: 5.4 next class.