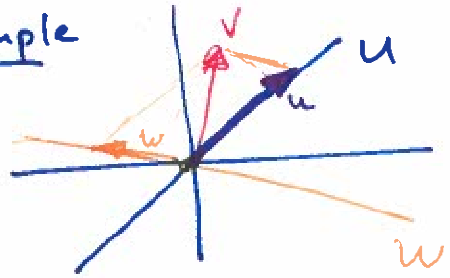
Today: rank of a (matrix) linear map.

Today: more on rank, and how to find it.

First, some examples, and more about linear maps.

Example



$\mathbb{R}^2$

Any two linear subspaces U, W of V such that

$$U \oplus W = V$$

(i.e., $U \cap W = \{0\}$ and $U + W = V$)

give rise to a projector:

$$P_U^W : V \to V$$ – projector onto U along W.

(exer: this works if and only if $V = U \oplus W$):

recall: (from HW) $V = U \oplus W \iff \forall v \in V$

exists unique $\exists!\ u \in U,\ w \in W$

s.t. $v = u + w$

this says: every vector has well-defined components along U and W.

Define $\boxed{P_U^W(v) = u}$ ← forget the W-component.

Then by definition, $\operatorname{Ker}(P_U^W) = W$

$\operatorname{Im}(P_U^W) = U$.

This is a main example of a linear map with non-trivial kernel.

Dimension formula

If $A: V \to W$ - linear map

then
$$\boxed{\dim(\ker A) + \underbrace{\dim(\operatorname{Im} A)}_{rk(A)} = \dim(V)}$$

Pf: $\ker(A) \subset V$

Start with $v_1, \dots, v_k$ - basis of $\ker(A)$.

Let $v_{k+1}, \dots, v_n$ be vectors in V needed to complete it to a basis of V

(basis extension Thm).

Then look at $A(v_{k+1}), \dots, A(v_n)$ - images of $v_{k+1}, \dots, v_n$.

Exer: They are lin. indep. (in W) and span $\operatorname{Im}(A)$.

This completes the proof: this means,
$$rk(A) = \dim(\operatorname{Im}(A)) = n-k \quad (k = \dim(\ker(A))).$$

More about matrices

Upshot: A vector space of dim n over F is identified with F^n by a choice of basis.

We have: $F^n = \{(a_1, \dots, a_n) \mid a_i \in F\}$.

standard basis $e_1, \dots, e_n$

$e_1 = (1, 0 \dots 0), \dots \quad e_n = (0, \dots 0, 1)$ notation only makes sense in F^n

Let V be a vector space over F, $\dim V = n$

Let $v_1, \dots, v_n$ be a basis of V.

Now can identify V with F^n:

more precisely, make an isomorphism $A: V \to F^n$.

map (transformation) preserving the properties of the kinds of objects you are working with

(a bijective linear operator)

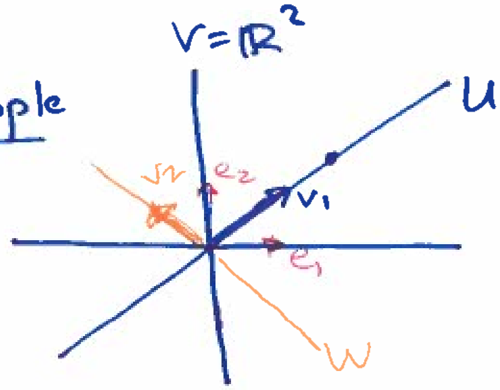
one-to-one and onto

Define: $A(v_i) = e_i$ for $i = 1, \dots, n$

This means, for $v = \lambda_1 v_1 + \dots + \lambda_n v_n$, $A(v) = (\lambda_1, \dots, \lambda_n)$ – the coordinates of v with respect to the basis $(v_1, \dots, v_n)$.

Matrices of linear transf. of V depend on the choice of basis.

Example



Want a convenient basis for writing a matrix for P_U^W

note: writing this matrix in $\{e_1, e_2\}$ basis is hard.

(impossible because equations of U, W are not given)

$v_1 \in U$ – a basis vector

$v_2 \in W$ – a basis vector

$\{v_1, v_2\}$ – basis of V.

In this basis, P_U^W has the matrix: $\left(\begin{array}{c|c} 1 & 0 \\ 0 & 0 \end{array}\right)$

v_1 goes to itself

In general, if we have a projector P_U^W, if we take a basis of V that is a union of a basis of U and a basis of W, the matrix of P_U^W will look like this:

$$
\begin{array}{cc}
 & \overbrace{}^{k} \\
k & \left(\begin{array}{ccc|ccc}
1 & & 0 & 0 & & 0 \\
 & \ddots & & 0 & & 0 \\
0 & & 1 & 0 & & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) n
\end{array}
$$

Id_U

$n-k$ rows of 0's

n

$n-k$ cols of 0's.

$\dim V = n$

$\dim U = k$

$\dim W = n-k.$

(because on U, P_U^W acts as the identity of U, and W goes to 0).

Rank of a matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

– matrix of a linear map: $F^n \to F^m$ (F is a field, $a_{ij} \in F$).

$\mathrm{rk}(A) = \dim(\mathrm{Im}(A)) = \dim$ (subspace of F^m spanned by the columns of A)

"column rank"

Magic fact: define row rank of A as $\dim$ (subspace of F^n spanned by the rows of A).

Magic: $\boxed{\text{row rank} = \text{column rank}}$

Why:
- read the book
- prettier explanation but it uses a new operation: use dot product.

$v = (\lambda_1, \dots, \lambda_n)$, $w = (\mu_1, \dots, \mu_n)$

$v \cdot w = \lambda_1\mu_1 + \dots + \lambda_n\mu_n$.

Not linear, but if one vector is fixed, it is linear as a function of the other vector.

$$A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} r_1 \cdot x_1 \\ \vdots \\ r_n \cdot x_n \end{pmatrix}$$

where $r_i = (a_{i1}, \dots, a_{in})$ ith row of A.

Then $\mathrm{Ker}(A) = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : r_i \cdot X = 0 \text{ for all } i = 1, \dots, n \right\}$

X = {vectors perpendicular to all $r_1, \dots, r_n$}.

$= L(r_1, \dots, r_n)^\perp$

linear space spanned by rows.

fact.

$$\underbrace{\overbrace{\mathrm{Ker}(A)^\perp}^{L(r_1,\dots,r_n)}}_{\dim = \dim V - \dim(\mathrm{Ker}(A))} \oplus \mathrm{Ker}(A) = V.$$

$\dim = \dim V - \dim(\mathrm{Ker}(A)) = \mathrm{rk}(A)$ done.