

Last time

vector spaces over a field F (finite-dim.)

$A: V \rightarrow W$ - linear transformation

"isomorphic"
(will define
in a
minute)

$\begin{matrix} \text{is} \\ \mathbb{F}^n \end{matrix}$ $\begin{matrix} \text{is} \\ \mathbb{F}^m \end{matrix}$

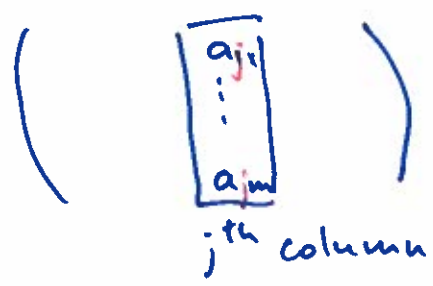
$\{e_1, \dots, e_n\}$
 \downarrow
 $\{e'_1, \dots, e'_m\}$

use the standard bases in both $\mathbb{F}^n, \mathbb{F}^m$

Then we can write A as a matrix:
(a matrix for A)

The j^{th} column of A is the image of the basis vector e_j :

$$e_j \xrightarrow{A} a_{j1} e'_1 + a_{j2} e'_2 + \dots + a_{jm} e'_m \quad \text{for } 1 \leq j \leq n$$



Def: If V, W are vector spaces over F , an isomorphism $A: V \rightarrow W$ is a linear transformation that has an inverse (as a function),

i.e. there exists $B: W \rightarrow V$ s.t.
"which means" $B \circ A = Id_V$ $V \xrightleftharpoons[B]{A} W$
 $A \circ B = Id_W$

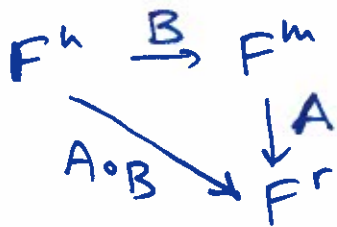
Exer: (h.w.) if such an inverse function B exists, it has to be a linear transformation!

If an isomorphism between V and W exists, they are called isomorphic vector spaces.

Will prove later today that any n -dimensional vector space is isomorphic to F^n .

Because of this, all linear trans. of finite-dim. vector spaces can be studied using matrices.

How to compose linear transformations?



Composition of lin. trans. is a lin. trans.

What is the matrix of the composition?

Answer: matrix multiplication (tricky to define).

Example: a lin. trans. $F^n \rightarrow F$ (no composition yet!)

$(a_{11} \dots a_{1n}) \leftarrow$ one row
 \uparrow
 image of e_i

When we apply it to a vector in F^n , we can write it as:

$$(a_{11} \dots a_{1n}) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n}_{\substack{\mathbb{P} \\ F}}$$

The simplest matrix product

\uparrow column vector in F^n
 (we always write vectors as columns!)

This does give a map from F^n to F .
 Check it agrees with A : only need to check it does the right thing to the basis vectors.

Example: $\mathbb{R}^2 \xrightarrow{(i\ j)} \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$
 $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix}$

↑
 "permutation matrix"
 multiplying by this matrix has permuted the columns of A!

In the other order: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
 $= \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$ ← permuted the rows of A!

Note: $AB \neq BA$!

Let us prove that it does correspond to the composition of linear transformations.

$F^n \xrightarrow{B} F^m \xrightarrow{A} F^r$
 $\{e_j\} \quad \{e'_j\} \quad \{e''_j\}$

Need to show, that $A \circ B$ agrees with the one that has matrix AB

on the basis vectors e_j

$e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ (jth spot) $\mapsto \begin{pmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{pmatrix} = b_{1j} \cdot e'_1 + \dots + b_{mj} \cdot e'_m$
 ↑ decomposition in the basis of F^m
 ↑ jth column of the matrix B (in F^m)

Apply A to this.

Then

$$A \circ B(e_j) = A(b_{1j}e_1' + \dots + b_{mj}e_m')$$

$$= b_{1j}A(e_1') + \dots + b_{mj}A(e_m')$$

A is linear

$$= b_{1j} \begin{pmatrix} a_{11} \\ \vdots \\ a_{r1} \end{pmatrix} + b_{2j} \begin{pmatrix} a_{12} \\ \vdots \\ a_{r2} \end{pmatrix} + \dots + b_{mj} \begin{pmatrix} a_{1m} \\ \vdots \\ a_{rm} \end{pmatrix}$$

\uparrow 1st column of A (vector of length r) \uparrow mth column of A.

$$= \begin{pmatrix} b_{1j}a_{11} + b_{2j}a_{12} + \dots + b_{mj}a_{1m} \\ b_{1j}a_{21} + \dots + b_{mj}a_{2m} \\ \vdots \\ b_{1j}a_{r1} + \dots + b_{mj}a_{rm} \end{pmatrix}$$

recognize this as the j^{th} column of AB:

the i^{th} entry is the product of the i^{th} row of A with the j^{th} column of B

Consider the product matrix AB, apply it to e_j

$$\begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{r1} & \dots & c_{rn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} c_{1j} \\ \vdots \\ c_{rj} \end{pmatrix}$$

\uparrow j^{th} spot \equiv \uparrow row i , j^{th} entry \equiv \uparrow j^{th} column of AB.

