Last time

vector spaces over a field F (finite-dim.)

- $A : V \to W$ — linear transformation

$V \cong F^n$, $\quad W \cong F^m$

"isomorphic" (will define in a minute)

$\{e_1, \dots, e_n\}$ $\quad \{e'_1, \dots, e'_m\}$

use the standard bases in both F^n, F^m

Then we can write A as a matrix! (a matrix for A)

The j^{th} column of A is the image of the basis vector e_j:

$$e_j \xmapsto{A} a_{j1} e'_1 + a_{j2} e'_2 + \dots + a_{jm} e'_m \quad \text{for } 1 \le j \le n$$

$$\begin{pmatrix} & \boxed{\begin{matrix} a_{j1} \\ \vdots \\ a_{jm} \end{matrix}} & \end{pmatrix}$$

j^{th} column

Def: If V, W are vector spaces over F, an isomorphism $A : V \to W$ is a linear transformation that has an inverse (as a function),

i.e. there exists $B : W \to V$ s.t.

"which means"

$$B \circ A = Id_V \qquad V \underset{B}{\overset{A}{\rightleftarrows}} W$$

$$A \circ B = Id_W$$

Exer: (h.w.) if such an inverse function B exists, it has to be a linear transformation!

If an isomorphism between V and W exists, they are called isomorphic vector spaces.

Will prove later today that any n-dimensional vector space is <u>isomorphic</u> to F^n.

Because of this, <u>all</u> linear transf. of finite-dim. vector spaces can be studied using matrices.

<u>How to compose linear transformations?</u>

$$\begin{tikzcd} F^n \arrow[r, "B"] \arrow[dr, "A\circ B"'] & F^m \arrow[d, "A"] \\ & F^r \end{tikzcd}$$

composition of lin. transf. is a lin. transf.

What is the matrix of the composition?

<u>Answer</u>: <u>matrix multiplication</u> (tricky to define).

<u>Example</u>: a lin. transf: $F^n \to F$ (no composition yet!)

$(a_{11} \dots a_{1n}) \leftarrow$ <u>one</u> row

a_{11}: image of e_1 . . .

When we apply it to a vector in F^n, we can write it as:

$$(a_{11} \text{--} a_{1n}) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \overset{\text{def}}{=} \underbrace{a_{11}x_1 + a_{12}x_2 + \text{--} + a_{1n}x_n}_{\in F}$$

the simplest matrix product

column vector in F^n

(we always write vectors as columns!)

This does give a map from F^n to F.

Check it agrees with A: only need to check it does the right thing to the basis vectors.

Example: $\mathbb{R}^2 \xrightarrow{\begin{pmatrix}0&1\\1&0\end{pmatrix}} \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} \boxed{a_{11}} & \boxed{a_{12}} \\ \boxed{a_{21}} & \boxed{a_{22}} \end{pmatrix}}_{A} \cdot \overset{B}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \boxed{a_{12}} & \boxed{a_{11}} \\ \boxed{a_{22}} & \boxed{a_{21}} \end{pmatrix}$$

"multiplying by this matrix has permuted the columns of A!

"permutation matrix"

In the other order: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$= \begin{pmatrix} \boxed{a_{21} \quad a_{22}} \\ \boxed{a_{11} \quad a_{12}} \end{pmatrix}$$ ← permuted the rows of A!

Note: $AB \neq BA$!

Let us prove that it does correspond to the composition of linear transformations.

$$F^n \xrightarrow{B} F^m \xrightarrow{A} F^r$$
$$\{e_j\} \qquad \{e_j'\} \qquad \{e_j''\}$$

Need to show, that $A \circ B$ agrees with the one that has matrix AB

on the basis vectors e_j

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{pmatrix} = b_{1j} \cdot e_1' + \dots + b_{mj} e_m'$$

j^{th} spot

↑ decomposition in the basis of F^m

↑ j^{th} column of the matrix B (in F^m)

Take $e_j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← j-th place

$$(a_{11} \; \cdots \; a_{1n}) \cdot \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \cdot a_{11} + \cdots + 1 \cdot a_{1j} + 0 \cdots + 0$$

$$= a_{1j} \leftarrow \text{image of } e_j$$

so this formal product gives the right answer for each basis vector.

Def of matrix product

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ b_{12} & & \\ \vdots & & \\ b_{1n} & & \end{pmatrix}$$

$m \times n$ matrix

$$\begin{array}{ccc} F^n & \xrightarrow{B} & F^m \\ & \searrow & \downarrow A \\ & & F^r \end{array}$$

$$\underset{A}{\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{r1} & \cdots & a_{rm} \end{pmatrix}} \cdot \underset{B}{\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}}$$

$$= \begin{pmatrix} \boxed{\text{1st row} \cdot \text{1st col}} & \boxed{\text{1r} \times \text{2c}} & \boxed{} \\ & & \\ \boxed{} & & \boxed{} \end{pmatrix}$$

1st row × last column

last row × 1 column of B

last row of A × last col. of B

Apply A to this.

Then

$$A \circ B(e_j) = A(b_{1j} e_1' + \cdots + b_{mj} e_m')$$

$$= b_{1j} A(e_1') + \text{---} + b_{mj} A(e_m')$$

A is linear

$$= b_{1j} \begin{pmatrix} a_{11} \\ \vdots \\ a_{r1} \end{pmatrix} + b_{2j} \begin{pmatrix} a_{12} \\ \vdots \\ a_{r2} \end{pmatrix} + \text{---} + b_{mj} \begin{pmatrix} a_{1m} \\ \vdots \\ a_{rm} \end{pmatrix}$$

1st column of A (vector of length r)

mth column of A.

$$= \begin{pmatrix} b_{1j} a_{11} + b_{2j} a_{12} + \text{---} + b_{mj} a_{1m} \\ b_{1j} a_{21} + \text{---} + b_{mj} a_{2m} \\ \boxed{b_{1j} a_{r1} + \text{---} + b_{mj} a_{rm}} \end{pmatrix}$$

recognize this as the jth column of AB:

the ith entry is the product of the ith row of A with the jth column of B

consider the product matrix AB, apply it to e_j

c_{ij}

$$\underbrace{\begin{pmatrix} c_{11} & \text{---} & c_{1n} \\ & & \\ c_{r1} & \text{---} & c_{rn} \end{pmatrix}}_{AB} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \text{row 1, } j\text{th entry} \\ \vdots \\ j\text{th entry of row } r \end{pmatrix} = \boxed{j\text{th column of } AB.}$$

jth spot