Homework 9: Eigenvalues 2: Jordan normal form, self-adjoint linear operators
Due Monday April 17 at 10pm on Canvas.

1. (a) Problem 10.1 from Janisch
   (b) Find A^5 for the matrix A from this problem.
2. (a) Let $A : V \to V$ be a self-adjoint linear operator on a vector space V. Prove that if U is an invariant subspace for A, then $U^\perp$ is also invariant under A.
   (b) Problem 10.3 from Jänisch
3. Let J_λ be the 3×3 Jordan block with the eigenvalue λ:
$$J_\lambda = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$
   Find J_λ^2, J_λ^3 and J_λ^4.
   Hint: It might help to use the binomial formula for matrices.
4. (a) Let $A : V \to V$ be a linear operator that has only one eigenvalue in V, with multiplicity $n = \dim(V)$. Prove that the Jordan normal form of A is a single block if and only if V has a basis $v_1, \dots, v_n$ that satisfies: $Av_1 = \lambda v_1$ (i.e., v_1 is an eigenvector), and $Av_k = v_{k-1} + \lambda v_k$ for $k = 2, \dots, n$.
   (b) Let $\lambda \in \mathbb{R}$ be fixed, and let V be the n-dimensional subspace of the space of smooth functions spanned by the functions of the form $x^k e^{\lambda x}$, $k = 0, \dots, n-1$. Find a basis of V that satisfies the property from part (a) for the differentiation linear operator $D(f) = f'$ that acts on V.
   Remark. This explains the appearance of the functions $x^k e^{\lambda x}$ when the characteristic equation has multiple roots, when you solve homogeneous differential equations with constant coefficients.
5. Let $a_0, \dots, a_{n-1}$ be scalars. Find the charactersitic polynomial of the

matrix

$$\begin{vmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & \dots & 0 & 1 & 0 & -a_{n-2} \\ 0 & \dots & & \dots & 1 & -a_{n-1}) \end{vmatrix}.$$

Hint. Look at the last problem of Homework 6.

Remark. This shows that any polynomial arises as the characteristic polynomial of some matrix (in case you were wondering). This matrix is called the *Frobenius companion matrix* for a given polynomial.