Review topics for the Math 223 final exam.

This is a rough list of the things you should definitely be able to do. I think the best way to use this review sheet is to try to recall or come up with an example for each concept or technique listed below. Then try to solve that example, and if you cannot (or could not come up with one to begin with), then re-read the corresponding section in the book. It is very important to try to think of all these concepts without opening the book, at first. Copies of this review sheet will not be allowed at the exam.

A good way to practice is to complete the "Test" sections in the book.

In addition, you are responsible for Section 2.5 (not covered by the "Test" sections), and Gaussian elimination (Section 7.5 but without column operations); reduced row echelon form (see notes for Feb. 13 on the website).

1. Vector spaces, bases, linear dependence and independence

- The notions of a vector space and a linear subspace. Determining whether a given subset of a vector space is a linear subspace.
- Deciding whether a given vector belongs to a given subspace.
- The notions of a basis and dimension.
- Deciding whether a given set of vectors is linearly independent, and find a subset that forms a basis of their linear hull.
- Finding bases of vector spaces.
- Fields: the notion of a field; the complex numbers; the field of p elements.

2. Linear transformations and matrices

- The notion of a linear transformation, and how to associate a matrix with it (given a choice of a basis).
- An example: projector onto a linear subspace U along a subspace W.
- Rank of a linear transformation and rank of a matrix
- The kernel of a linear transformation
- The formula $\dim(\ker(A)) + \mathrm{rk}(A) = \dim(V)$ (and need to understand why it holds).
- Finding the (reduced) row echelon form of a matrix via elementary row operations
- Using elementary operations to determine rank of a matrix

3. Systems of linear equations

- Writing an (augmented) matrix associated with a system of linear equations, bring the matrix to RREF (reduced row echelon form), and how that gives a solution to the given system of equations.
- Pivot columns and pivot variables
- Finding a basis for the kernel of a linear transformation
- The inverse matrix; when does it exist?
- Finding the inverse of a matrix using elementary row operations.

4. Determinants

- Computing the determinant of a matrix using decomposition along one row or column.
- Cramer's rule.

- The adjugate matrix; the formula for an inverse of a 2×2 -matrix.
- Determinant of an upper-triangular matrix; determinant of a block-triangular matrix.
- Determinant of a product of matrices.

5. Euclidean spaces

- Inner products
- Cauchy-Schwarz inequality
- The main point you need to understand well: if $\{v_1, \dots, v_n\}$ is an orthonormal basis of V, then the coordinates of any vector $x = (x_1, \dots, x_n)$ of any vector $x \in V$ with respect to this basis can be obtained as inner products: $x_i = \langle x, v_i \rangle$.
- Gram-Schmidt orthonormalization process (you need to be able to find an orthonormal basis of a given linear space).
- Orthogonal complements and orthogonal projectors.
- Orthogonal linear transformations.

6. Eigenvalues and eigenvectors

- The definition of eigenvalues and eigenvectors for a linear operator $A : V \to V$.
- Linear independence of eigenvectors corresponding to distinct eigenvectors
- Finding eigenvalues: the roots of the characteristic polynomial.
- You need to be able to find eigenvalues and eigenvectors of a given linear operator (assuming the roots of the characteristic polynomial are easy to find).
- Diagonalization of a matrix with distinct eigenvalues.
- Changing a basis; transition matrix. For this, you need to know two formulas: if you have two bases $\{v_i^{\text{old}}\}$ and $\{v_i^{\text{new}}\}$, and C is the transition matrix (its j-th column is the coordinates of the vector $\{v_j^{\text{new}}\}$ with respect to the old basis), then for any $x \in V$, if $x^{\text{old}} = (x_1, \dots, x_n)^{\text{old}}$ is the tuple of its coordinates with respect to the old basis, and $(x_1, \dots, x_n)^{\text{new}}$ are the coordinates of the same point with respect to the new basis, then $x^{\text{old}} = Cx^{\text{new}}$. For a linear operator $A : V \to V$, if A^{old} is its matrix with respect to the old basis, and A^{new} is its matrix with respect to the new basis, then $A^{\text{new}} = C^{-1}A^{\text{old}}C$.

  Given a matrix A, you need to be able to find a transition matrix such that $C^{-1}AC$ is diagonal (if such a C exists).
- The general case: you need to know what happens when the characteristic polynomial of A has multiple roots (meaning, some eigenvalues have algebraic multiplicity greater than 1). You need to know the statement of Jordan Normal Form and some examples of what it might look like.
- You need to know how to apply the above theorems to find powers of a given linear operator.
- Self-adjoint linear operators. The main result about them: for a self-adjoint linear operator, there exists an *orthonormal* basis consisting of its eigenvectors.

For matrices, this means: if you start with a symmetric matrix A (with respect to an orthonormal basis), there exists an *orthogonal* matrix P such that $P^{-1}AP$ is diagonal.

You need to be able to find this P (called the *principal axes transformation*). (See "recipe" on p. 168 of the textbook).

You should expect a problem very similar to Problem 10.1 in the textbook.

- The spectral theorem, and spectral decomposition of a self-adjoint linear operator (see p. 167).