Extra credit assignment: harder problems.

You can hand in any number of these problems by 11:59pm on April 23 (on Canvas). Each complete problem adds 2% to your total term mark (except for Problem 2, which is very easy, and only adds 0.5%).

1. **Vandermonde Determinant.** The goal of this problem is to compute the determinant of the matrix A defined by $a_{ij} = x_i^j$, for $i, j = 0, \ldots, n$, where $x_0, \ldots, x_n$ are variables (so it is an $(n+1) \times (n+1)$ -matrix.

   The goal of this problem is to prove that $\det(A) = \prod_{0 \leq i < j \leq n}(x_i - x_j)$.

   (a) Compute the 2×2 and 3×3 Vandermonde determinants.

   Hint. We actually did this in lecture.

   (b) Use column operations to make the first row have the form $1\,0\,\ldots 0$. Record the resulting matrix.

   (c) Now use induction to prove the result.

   Remark. Many other proofs exist. One of my favourite ones uses the properties of polynomials: fix the values of all the variables except for x_0, and think of x_0 as a variable. Now if you plug any of the fixed values x_i for x_0, the determinant clearly becomes 0. Then (by the properties of polynomials that you will study in Math 323) the expression $x_0 - x_i$ has to divide the determinant (viewed as a polynomial in the x_i). Since you could swap rows, this applies to every expression $x_i - x_j$. Now just comparing the degrees and leading coefficients of these polynomials, we obtain the result.

2. **Operator calculus.** Let $A : V \to V$ be a linear operator on a vector space V over a field F (we are not assuming that V is finite-dimensional in this problem). We define the powers $A^n : V \to V$ as the composition of A with itself n times: $A^n(v) = A(A(..(Av))..)$. Then given a polynomial $p(x) = a_n x^n + \cdots + a_0$, where $a_i \in F$, we can define $p(A) : V \to V$ to be the linear operator $p(A) = a_n A^n + \cdots + a_0 I$, where $I : V \to V$ is the identity. Suppose that v is an eigenvector for A with eigenvalue λ, i.e, $Av = \lambda v$, and $v \neq 0$.

   (a) Prove that v is an eigenvector for A^n with eigenvalue λ^n.

   (b) Prove that v is an eigenvalue for $p(A)$ with eigenvalue $p(\lambda)$.

3. **Linear recurrences.** Let V be the complex vector space of all sequences $\bar{x} = (x_0, x_1, \dots, x_n, \dots)$ with $x_i \in \mathbb{C}$ (it is infinite-dimensional). We say that a sequence $\bar{x} \in V$ satisfies a *linear relation of degree* k if there exist coefficients $c_0, \dots, c_{k-1} \in \mathbb{C}$ with $c_0 \neq 0$ such that $x_{n+k} = \sum_{i=0}^{k-1} c_i x_{n+i}$ for all $n \geq 0$. The goal of this problem is to explore how to find all the sequences satisfying a given linear recurrence relation of degree k. We define the *characteristic polynomial* of a linear recurrence relation by

$$p(t) = t^k - \sum_{i=0}^{k-1} c_i t^i.$$

   (a) Write down in this form the linear relation defining the Fibonacci sequence. What is its degree?

   (b) Explain why we require $c_0 \neq 0$.

   (c) Let $L : V \to V$ be the *left shift operator*: $L(x_0, x_1, \dots) = (x_1, x_2, \dots)$. Prove that a sequence $\bar{x} \in V$ satisfies a linear recurrence relation if and only if it lies in the kernel of the linear operator $p(L) : V \to V$ (see the above problem for the meaning of $p(L)$).

   (d) Prove that a sequence satisfying a linear recurrence of degree k is determined by the k initial values $x_0, \dots, x_{k-1}$. Conclude that $\ker p(L)$ has dimension k.

   (e) Assuming that $p(t)$ has k distinct roots $\lambda_1, \dots, \lambda_n$, find a basis for $\ker p(L)$.

   (f) Let $(F_0, F_1, \dots, F_{k-1})$ be any numbers. Show that the system of k equations $\sum_{i=0}^{k-1} A_i \lambda_i^j = F_j$ $(1 \leq j \leq k)$ in the unknowns A_i has a unique solution.

   (g) Show that for any recurrence relation of degree k, any initial k-tuple of values extends to a unique solution of the recurrence relation.

   (h) Find a non-recursive formula for the n-th Fibonacci number.

4. **Applications of Cauchy-Schwarz inequality.**

   (a) Prove that if the series $\sum_{n=1}^{\infty} |a_n|^2$ and $\sum_{n=1}^{\infty} |b_n|^2$ converge, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.

   (b) If $a_1 + a_2 + \cdots + a_n = n$ show that $a_1^4 + \cdots + a_n^4 \geq n$.
   Hint: Apply Cauchy-Schwarz twice.