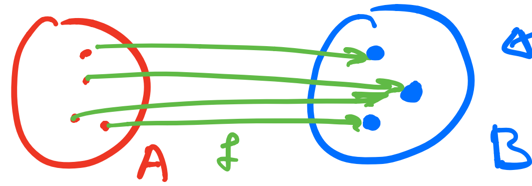
Math 223, Diagnostic quiz.

Let A be a set of red dots, and B be a set of blue dots. We call a function $f : A \to B$ *nice* if for every blue dot there is a red dot that maps to it: for every blue dot $b \in B$ there is a red dot $a \in A$ such that $f(a) = b$.

1. Draw a picture illustrating the concept of a nice function.

   Solution.

   it is a function: exactly one arrow from each red dot.

   no blue dot is lonely

2. Let us call a blue dot *lonely* if there is no red dot that maps to it. State the converse, contrapositive, and negation of the following statement:

   > If a function is nice, then no blue dot is lonely.

   Label each of your statements as True/False (no proof needed, but do not guess).

   Solution. Let P be the statement "the function is nice", and Q be the statement "no blue dot is lonely" (which means, "there does not exist a lonely blue dot"). Our statement says, $P \Rightarrow Q$ (it is an *implication*: P implies Q, or "if P, then Q".

   The *converse* is: $Q \Rightarrow P$, which reads,

   > If no blue dot is lonely, then the function is nice.

   Here it happens to be true, because by definition of a nice function, in fact, we have $P \Leftrightarrow Q$.

   The *contrapositve* states: "Not Q implies not P". It is always equivalent to the original statement that P implies Q (think about this!). Here the "Not P" says: "the function is NOT nice", and the "Not Q" says: "There exists a lonely blue dot" (or, colloquially, *some* blue dot is lonely) (be careful here about negating the statement which has a quantifier). Putting it together, the contrapositive states:

   > If there exists a lonely blue dot, then the function is not nice.

This statement is True (it is equivalent to the original one).

Finally, the *negation* of $P \Rightarrow Q$ is **NOT an implication at all!** The statement "not $(P \Rightarrow Q)$" says, simply, that P holds but Q doesn't, so the negation of $P \Rightarrow Q$ is "P and not Q" (think about it and construct a truth table to convince yourself!). In our situation, it would say:

> The function is nice and there exists a lonely blue dot.

This statement is False (the original statement was True).

Remark: our definition of a 'nice' function in fact is the definition of a surjective function.