

A note about the use of "such that" and other issues with quantifiers.

This note is about correct use of the expression "such that" (and more generally, about understanding statements that involve quantifiers).

Can you spot a mistake in the following example:

Question: Negate the statement

$$(1) \quad \exists x \in \mathbb{R}, \text{ s.t. } \forall y \in [x - 1, x + 1], y \geq -x.$$

Answer:

$$(2) \quad \forall x \in \mathbb{R}, \text{ s.t. } \exists y \in [x - 1, x + 1], y \leq -x.$$

Solution: This answer is almost right, except for the use of "s.t.", but unfortunately this "s.t." in the wrong place makes the statement meaningless. If not sure why, please read on.

When we talk about people, we can use the pronoun "who" (or "whom, whose") when we want to refer to them. For example, we can say the phrase

(3) "There exists a student in the class all of *whose* friends are older than him/her"

This statement may be true or false, but this is a statement (from the point of view of logic). When one wants to make a similar statement about inanimate objects (rather than humans), the words "such that" are a good replacement for all the forms of "who". For example, we can make the statement

(4) "There exists a tree in Stanley park *such that* all the neighbouring trees are at least as tall as this tree."

We could have phrased this as "there exists a tree in Stanley Park all of whose neighbours are at least as tall", but it would not sound quite right, though it would be the same statement.

Note that the words "such that" are necessary in this sentence, otherwise it would not make sense. In short, the words "such that" usually go together with the existential quantifier " \exists " – they specify the required property of the object whose existence we claim. For example, the statement

$$(5) \quad \exists x \in \mathbb{R}, \text{ s.t. } x^3 - x + 1 = 0.$$

is made correctly, but without "such that", it would have been hard to understand. The statement (1) in the example above is made correctly, too.

Now, let us try to make negations of all the statements above. I try to do it with wrong use of "such that" first.

Then we get the phrases: Negation of (3), if we do it without thinking, would sound as:

For all students in the class such that not of all whose friends are older than him/her...

The negation of (4), if we insist on using "such that", would sound like:

For all trees in Stanley Park such that not all the neighbouring trees are at least as tall as this tree ...

We notice that something funny happened – these do not sound like finished sentences; one expects a continuation, such as “so, what happens to all these students in the class such that not all of their friends are older?”, “what do you want to say about all the trees in Stanley park such that not all the neighbouring trees are at least as tall?”. Note that the wrong answer (2) to the opening question is constructed in exactly this way.

How can we fix these sentences? **By removing “such that”!** (and also sometimes in English sentences, “every” sounds better than “for all”). Consider instead the sentences

(6) Every student in the class has a friend who is not older than him/her.

(7) Every tree in Stanley Park has a neighbour which is not taller than this tree.

These are complete understandable sentences!

Now, here are some exercises:

1. Write the negation of (5), both in symbols and in words.
2. (a) Let S be the set of all students in the class, and for every student $s \in S$, denote by $F(s) \subseteq S$ the set of all friends of s .
Using this notation, write down the statement (3) in symbols.
(b) Let T be the set of all trees in Stanley Park. Make appropriate notation, and write down the statement (4) in symbols.
3. (a) Check whether the statement (6) is a correct negation of (3), and whether the statement (7) is a correct negation of (4).
(b) Write down the statements (6) and (7) using symbols (use the notation of the previous exercise).
4. Write down the correct negation of (1), using symbols.
5. Is (1) true?
6. (a) Do you have friends who are younger than you (or exactly the same age)? Based on just this information, can you conclude whether the statement (3) about our Math 220 class is true or false? Why or why not?
(b) Do you have friends who are older than you? Based on just this information, can you conclude whether the statement (3) about our Math 220 class is true or false? Why or why not?
7. Do you think the statement (4) is true? Why or why not? Do you have to go to Stanley Park and look at every tree and its neighbours in order to find out? If not, why not?

Big Hint: There are finitely many trees in Stanley Park, and among them, there has to be the shortest one.