(Epsilon, delta - Greek letters; we use them to denote small real numbers)

Worksheet 7: Quantifiers, Negation - Part 2.

1. Negate the statement: def. of "$f(x)$ is continuous at the point a".

sometimes not written

$$\forall \epsilon > 0 \exists \delta > 0, s.t. \forall x (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon).$$

" for every positive ε, exists a positive number δ, such that if the distance from x to a is less than δ, then the distance from $f(x)$ to $f(a)$ is less than ε. "

in words, no need to put "for all x"

2. Negate the statement:

$$\forall N > 0 \exists M > 0, s.t. \forall x (x > M \Rightarrow f(x) > N).$$

3. Write the statement '$\exists y_0 \in \mathbb{R}, \forall x \in \mathbb{R}\ x^2 - 2x + 3 \geq y_0$' in words. Is this statement true or false? Write its negation both in symbols and in words. If this y_0 exists, is it *unique*?

4. Let $f(x)$ be some function (from the real numbers to the real numbers). Do the statements:

$$\exists y \forall x\ f(x) \leq f(y)$$

and

$$\forall x \exists y\ f(x) \leq f(y)$$

mean the same thing? Explain in words what each of them means. For each of the statements, make an example of a function that makes it true, and an example that makes it false.

Negating (1):

$$\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0,$$

$$\left(|x-a| < \delta \not\Rightarrow |f(x) - f(a)| < \varepsilon\right)$$

$$\Leftrightarrow \exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0$$

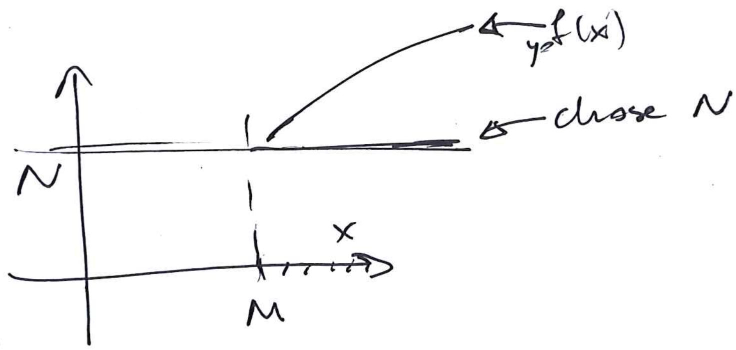
(generally, the domain of f)

$$\exists x \in \mathbb{R} \text{ s.t. } \underbrace{|x-a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon}_{\text{this is } \sim\left(|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon\right)}$$

Meaning of all this:

(2) in words:

"for every positive N, there is a positive M, such that if $x > M$ then $f(x) > N$"



Analysis of this statement:

"for all N" means: you have no control over N, someone gives you N.
Then you have to find M, such that for all $x > M$, you function $f(x)$ is bigger than N.

This is the definition of:

$$\boxed{f(x) \to \infty \text{ as } x \to \infty}$$

Question: how to say "monotonically increasing":

"$\forall x, y \in \mathbb{R}, \; ((x \le y) \Rightarrow (f(x) \le f(y)))$"

$\equiv$ "$\forall x \forall y \quad x \le y \Rightarrow f(x) \le f(y)$"

Abbreviation:

$$\boxed{\begin{aligned} \forall x, y &\equiv \forall x \forall y. \\ \exists x, y &\equiv \exists x \exists y \end{aligned}}$$

order doesn't matter

Negation of (2):

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$$\forall \epsilon > 0 \exists \delta > 0, s.t. \ \forall x (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon).$$

2. Negate the statement:

$$\forall N > 0 \exists M > 0, \underline{s.t.} \ \forall x (x > M \Rightarrow f(x) > N).$$

Watch where "s.t." goes!

$\exists N > 0 \ \underline{s.t.} \ \forall M > 0 \ \exists x \ (x > M \wedge f(x) \leq \underline{N})$

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Comment: why the quantifiers switch when you negate a statement?

Negating quantified statements

- All baby monsters are blue.

Negation: "not all" baby monsters are blue.

$\Leftrightarrow$ exists a baby monster who is not blue.

- Exists a positive number x satisfying the equation $x^2 - 3 = 0$

Negation: there is No positive number x s.t. $x^2 - 3 = 0$

$\Leftrightarrow \forall \; x > 0, \quad x^2 - 3 \neq 0.$

Careful: $\underbrace{\forall\ \varepsilon > 0}$ $\exists\ \delta > 0$

about positive numbers.

Do NOT write: $\underbrace{\exists\ \varepsilon < 0}$ $\forall\ \delta < 0$!!

very wrong

makes it a statement about negative numbers!

These statements are unrelated!

(3) in words: Exists $y_0 \in \mathbb{R}$ such that

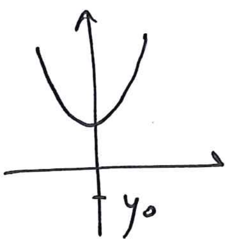
for every real x,

$$x^2 - 2x + 3 \geq y_0 .$$

True: Take $y_0 = 0$.

We have: $x^2 - 2x + 3 = (x-1)^2 + 2$

$$\geq 0$$



Yes, $y_0 = 0$ works!

(Any $y_0 \leq 2$ would work).

Negation: $\forall y_0 \in \mathbb{R}\ \exists x \in \mathbb{R}$ s.t.

$$x^2 - 2x + 3 < y_0$$

(False).

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(4) When quantifiers are different, the order really matters!

$$\exists y\ \forall x \quad f(x) \leq f(y)$$

exists y s.t. all other values of f are not greater than $f(y)$.

This says: $f(y)$ is a maximal value of f!

In the other order:

$$\forall x\ \exists y : \quad f(x) \leq f(y)$$

this y is allowed to depend on x.

For every x, exists y such that

$f(x)$ is not greater than $f(y)$

<u>Example</u> Take $f(x) = 5x$.

The statement (1) is False for this f.

Statement (2) is true: given x,

I can find a y s.t. $5x \leq 5y$.

We just cannot find the value y that works for <u>all x</u> at once!

Statement (2) ~~says~~ if the inequality were <u>strict</u>:

"f does not have a maximum"

& As is, (2) is <u>vacuously</u> true:

take $y = x$.

$\forall x \; \exists y$ s.t. $f(x) \leq f(y)$.

say $y = x$. get $f(\cancel{x}) = f(x)$ True.