Today: Quantifiers
Negating statements
(finish formal logic)

Next: Chapter 7

# Worksheet 6: Quantifiers, Negation.

Let us make some notation: let $S$ be the set of all students in the class, and for every student $s \in S$, denote by $F(s)$ the set of all friends of $s$. For a person $p$, let $a(p)$ be the age of $p$.

( **The previous worksheet:**

1. Using this notation, write in symbols the statement: "There exists a student in the class all of *whose* friends are older than him/her".

we can talk about students: $s \in S$ — set of all students in the class

a student; we can talk about age of s: $a(s)$ function

See next pages for the solutions.

2. Make similar notation and then write symbolically the statement "There exists a tree in Stanley park *such that* all the neighbouring trees are at least as tall as this tree."

3. Is this statement about Stanley park true or false?

**New problems**

Corrected worksheet posted separately.

1. Negate the statement in (1). ✓
2. Write the statement '$\exists y_0 \in \mathbb{R}, \forall x \in \mathbb{R}, x^2 - 2x + 3 \geq y_0$' in words. Is this statement true or false? Write its negation both in symbols and in words.
3. Write in words, then negate: Let $P$ be the set of all primes.

$$\forall N > 0, \exists p \in P, s.t. p > N, p + 2 \in P.$$

space ^

$$\forall N > 0, \exists p \in P \text{ s.t. } (p > N) \wedge (p+2 \in P)$$

4. Negate the statement:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x : |x - a| < \delta, |f(x) - f(a)| < \epsilon.$$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \, (|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$$

5. Negate the statement:

$$\forall N > 0, \exists M > 0, \forall x : x > M, f(x) > N.$$

$$\forall N > 0, \exists M > 0, \text{ s.t. } \forall x \in \mathbb{R}, (x > M \Rightarrow f(x) > N)$$

Question 1 from old worksheet

"There is a student in the class ...
exists

$\exists \; s \in S$

... all of whose friends ...

$\forall$ (want to talk about friends.
what friends?
- friends of s

Have to have a name for the set of friends of s - it is given: $F(s)$

$\forall x \in F(s)$, $\boxed{a(x) > a(s)}$ ← the friend x is older than the student s.

for every friend (call them x) of the student s

give a name to the "friend".

Answer: $\boxed{\exists s \in S, \underset{\text{optional}}{\text{s.t.}} \; \forall x \in F(s), \; a(x) > a(s)}$

Question: is this the same as:

$\exists \; s \in S \wedge x \in F(s), \quad a(x) > a(s)$ — No!

exists a student $s \in S$, and a friend x of s s.t. x is older than s

"a student has an older friend'
(but doesn't say anything about all their friends).

Wrong ways to try to say it:

$\exists\, s \in S \;\wedge\; \forall\, x \in F(s) \quad a(x) > a(s)$

exists a student in the class and all friends of this student are older.

$\exists\, s \in S\,,\; x \in F(s) \quad \underline{\forall\; a(x) > a(s)}$

almost ok

we want the stuff that goes after "such that" to be the defining property of the student we are looking for.

Need to have an element of a set right after the quantifiers:

$\forall\, x \in F(s)$ - fine

$\forall\,\underline{\ \ }\; a(x) > a(s)$ ← NOT OK

all who??

## Stanley Park:

let $T$ be the set of all trees in Stanley Park.

Need: to talk about neighbours of an individual tree.

so: make notation: $N(t)$ = the set of neighbours of a tree $t \in T$.

(this set depends on $t$!
so we call it $N(t)$ or $N_t$ ← talk about this later.)

(similar to $F(s)$ = set of friends of a student $s \in S$).

- we also need to talk about height of trees.
height is a function of a tree: a number assigned to $t$.
let $h(t)$ be the height of a tree $t$.

$$\text{"}\boxed{\exists\, t \in T \text{ s.t. } \forall\, p \in N(t),\ h(p) \geq h(t)}\text{"}$$

**Misstatements:**

e.g. $\text{"}\exists\, t \in T,\ \underbrace{\forall\, n \geq t}_{\text{what does this mean:}}\text{"}$

have to say what/who $n$ is. specify the set where it lives

"I mean all neighbours are at least as tall" ← need to introduce more notation.

the way I'd read this is:
"exists $t \in T$, such that for all $n$ that are bigger than $t$ ... ?".

Question 3 (from old worksheet):

Is the statement about St. Pk true?

$\exists\, t \in T$, s.t. $\forall\, p \in N(t),\ h(p) \geq h(t)$

Thinking of true/false?

Try to negate it:

- When negating: all quantifiers switch: $\exists \rightsquigarrow \forall$, $\forall \rightsquigarrow \exists$
- 'and's and "or"s switch
- the final 'simple statement' gets negated.

Formal negation:

need a 'such that' here.

$$\boxed{\forall\, t \in T \quad \exists\, p \in N(t) \quad \sim(h(p) \geq h(t))}$$

when negating formally, do not keep "such that"

"such that" goes with $\exists$.

when $\exists$ becomes $\forall$ when you negate, "such that" is harmful.

preliminary: For all trees every tree in Stanley Park, there is a neighbour of that tree, such that the neighbour is NOT at least as tall as that tree.

better: ~~For~~ Every tree in Stanley Park has a neighbour, that is shorter than that tree.

better: $\boxed{\text{Every tree in Stanley Park has a shorter neighbour.}}$ True/False?

False: if every tree had a shorter neighbour, there wouldn't be the shortest tree (we'd need infinitely many trees).

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We could prove the original statement:

"existence proof" – see Chapter 7:

Want to prove: Exists a tree in St. Pk s.t. all its neighbours are at least as tall.

- we see that the shortest tree in the whole park would qualify.
- The shortest tree exists because there are finitely many trees.

This is proof!

Worksheet
"New Problems"

#3 $\forall N > 0, \exists p \in P$ s.t. $(p > N) \wedge (p+2 \in P)$

For every positive number $N$, there exists a prime $p$ such that $p > N$ and $p+2$ is also prime.

This says: the pairs $(p, p+2)$ with both $p$ and $p+2$ prime exist with arbitrarily large $p$

! (Twin Prime Conjecture)

Humans do not know if it's T/F !

Negation:

formally: $\exists N > 0$ s.t. $\forall p \in P$,

$$\sim((p > N) \wedge (p+2 \in P))$$

$$\Leftrightarrow \exists N > 0 \text{ s.t. } \forall p \in P, (p \leq N) \vee (p+2 \notin P)$$

there is a number $N$, (positive) such that for every prime, either $p \leq N$ or $p+2$ is not prime.

Better: there is $N > 0$, such that for all primes $p$ that are bigger than $N$, $p+2$ is not prime.

mathematically (we know that there are inf. many primes)