Implication (Conditional Statements)
Direct proof (Theorems, Definitions...)

Last time: Conditional Statement:

\[ P \implies Q \]

"P implies Q" \rightarrow new statement

\[
\begin{array}{ccc}
P & Q & P \implies Q \\
T & T & T \\
T & F & F \\
F & T & T \\
F & F & T \\
\end{array}
\]

\( \text{\textup{a} - definition of the truth of } P \implies Q. \)

Discussion: Usually, if we say P implies Q, then P has something to do with Q. (we think, P causes Q.)

In formal logic, this does not have to be the case: if P has nothing to do with Q, but both are true, or P is false, then P \( \implies Q \) is still true.

Ex: If the Sun rises in the west (I have green hair.)

\( \implies \)

True: P false, Q false, P \( \implies Q \) - true.

Words: If P then Q, P therefore, Q... P implies Q
Worksheet 2: Conditional statements; divisibility

1. Decide whether the following statements are True or False; discuss why.

(a) If 2 is even, then 3 is odd. 
\[ P \quad \Rightarrow \quad T \quad \Rightarrow \quad T \quad \rightarrow \quad \text{True} \]

(b) If \( a \) is even, then \( a^2 \) is even.
\[ P \text{ needs proof (will do later)} \]
\[ \quad \text{open sentence depends on } a, \text{ whether T/F} \]

(c) 5 is even, therefore 3 is odd.
\[ P \quad \text{False} \quad \Rightarrow \quad Q \quad \text{False} \]
\[ \rightarrow \quad \text{True (again: } F \Rightarrow F) \]
\[ \text{(it's an example of (b), except } P \text{ is false).} \]

(d) 5 is even implies that 25 is even.
\[ \text{True (again: } F \Rightarrow F) \]
\[ \text{(but this is not relevant)} \]

(e) If a number \( a \) is even, then the number \( 2a + 3 \) is odd.
\[ a = 5 \]

(f) For any integer \( a \), the number \( 24a + 3 \) is odd.

2. Find the set of all positive divisors of 60.

3. Prove that for any integer \( n \), the number \( n(n + 1)(n + 2) \) is divisible by 6.

Convention: We imply that the statement should be true for all \( a \) if we say it is True.
About theorems:
some of them (the majority) are stated in the form $P \rightarrow Q$
Often, $P$ and $Q$ are open sentences, depending on some variables.

When we prove the theorem, we only argue about the variables that make $P$ True.
(Also, to prove a theorem, we need to prove it for all values of the variables for which $P$ is True.)

In our example (b) $P(a)$ $\quad Q(a)$

Theorem: If $a$ is even, then $a^2$ is even.

This means we need to prove it for all $a$.
sufficient to only consider even $a$
($a$ is odd $\implies$ anything)

First, need
Definition: What is an even number?
An even number is a number of the form $a = 2k$, where $k \in \mathbb{Z}$.
($k$ is an integer)

An odd number is a number of the form $a = 2k + 1$ for some $k \in \mathbb{Z}$. See below.

Even/odd applies only to integers.

$\mathbb{Z} = \text{(even)} \cup \text{(odd)}$

$\cup$ union of sets

$\{ n \in \mathbb{Z} : n = 2k \text{ for some } k \in \mathbb{Z} \} \cup \{ n \in \mathbb{Z} : n = 2k + 1 \text{ for some } k \in \mathbb{Z} \}$
What does \( a = 2k + 1 \) for some \( k \in \mathbb{Z} \) mean?

A few examples:

\[
\begin{align*}
\text{yes:} & \quad a = 10 \\
& \quad \text{can you find } k \in \mathbb{Z} \text{ such that } a = 2k? \\
& \quad k = 5 \\
\text{cannot find such } l \\
& \quad l \leq 4, 2l + 1 \leq 10 \\
& \quad 10 = 2 \cdot 4 + 1 \quad \text{False} \\
& \quad 10 = 2 \cdot 5 + 1 \quad \text{False} \\
& \quad l > 5, 2l + 1 > 10
\end{align*}
\]

For any \( a \), only one of these questions has the answer "yes."

If the first question has "yes" answer, then \( a \) is even.

If the second one has "yes", then \( a \) is odd.

Question: where do we name our number?

If I talk about a number, have to give it a name: \( a \), or \( n \), or \( k \), ...

"A number \( n \) is even if there is an integer \( k \) such that \( n = 2k. \)"

...new name for a new number.
Proof of our Theorem:

Then: If \( a \) is even then \( a^2 \) is even.

Let \( a \) be an even integer. \( \Rightarrow \) make \( P(a) \) True

Then there is \( k \in \mathbb{Z} \) such that 
\[ a = 2k \]

Then \( a^2 = (2k)^2 = 4k^2 \)
\[ = 2 \cdot (2k^2) \]

Then \( a^2 = 2\ell \) where \( \ell = 2k^2 \) - an integer!

as required to prove

How to get there: Last line: want to have

"Then \( a^2 \) is even"

this means, I want to show

that there is an integer \( \ell \)

such that \( a^2 = 2 \cdot \ell \)
Worksheet 2: Conditional statements; divisibility

1. Decide whether the following statements are True or False; discuss why.

(a) If 2 is even, then 3 is odd.

(b) If $a$ is even, then $a^2$ is even.

(c) 5 is even, therefore 3 is odd.

(d) 5 is even implies that 25 is even.

(e) If a number $a$ is even, then the number $2a + 3$ is odd.

\[
\text{Let } a \text{ be even. Then } a = 2k \text{ for some } k \in \mathbb{Z}.
\]

\[
\text{Then } 2a + 3 = 2(2k) + 3 = 4k + 3 = 2 \cdot (2k + 1) + 1
\]

(f) For any integer $a$, the number $24a + 3$ is odd.

3. Find the set of all positive divisors of 60.

2. Prove that for any integer $n$, the number $n(n + 1)(n + 2)$ is divisible by 6.

Want: $2a + 3$ is odd, so want to express $2a + 3$ as $2 \cdot \ell + 1$

Complaint: Theorem is true, but its assumption was unnecessary! $2a + 3$ is odd for any integer $a$, not just even $a$.

Our (e) is a special case of this general statement.

OK to prove a more general statement.
Divisibility

Def: Let $a,b$ be integers, $b \neq 0$.

We say $b \mid a$ ("$b$ divides $a$") if $a = bk$ for some integer $k$.

($b$ is called a divisor of $a$)

Hint for #3: from worksheet: (we will prove next time. Think about it!)

Lemma: $n \in \mathbb{Z}$. If $2 \mid n$ and $3 \mid n$, then $6 \mid n$. 

* a theorem used in a proof of a bigger theorem.
Worksheet 2: Conditional statements; divisibility

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   (e) If a number \( a \) is even, then the number \( 2a + 3 \) is odd.

   (f) For any integer \( a \), the number \( 24a + 3 \) is odd.

2. Find the set of all positive divisors of 60.

\[
\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\} = \{n \in \mathbb{Z} : n \mid 60\}
\]

3. Prove that for any integer \( n \), the number \( n(n + 1)(n + 2) \) is divisible by 6.

Proof in 3 steps:

Step 1: Prove it is divisible by 2
Step 2: Prove it is divisible by 3
Step 3: Prove the lemma (see prev. page)