Office hours: Today: 3-4 pm, Fridays 11-12.
(generally: Tue: 3:30 - 4:30)

Today: 2.3, Chapter 4. 4.1-4.4

Today: Implication (Conditional statements)
Direct proof (Theorems, Definitions...)

Last time: Conditional statement:
P, Q - statements

$$\boxed{\begin{array}{c} P \Rightarrow Q \\ \text{"P implies Q"} \end{array}}$$ ← new statement

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

← definition of the truth of $P \Rightarrow Q$.

Discussion: usually, if we say P implies Q, then P has something to do with Q.
(we think, P causes Q).
In formal logic, this does not have to be the case: if P has nothing to do with Q, but both are true or P is false, then
$P \Rightarrow Q$ is still true.

Ex: If the (sun rises in the west) (I have green hair.)
$\underset{P}{}$ $\quad\quad$ ← Q

True: P false, Q false, $P \Rightarrow Q$ - true.

Words:
If P then Q, P therefore, Q...
P implies Q

Worksheet 2: Conditional statements; divisibility

1. Decide whether the following statements are True or False; discuss why.

   (a) If $\underbrace{\text{2 is even}}_{P}$, then $\underbrace{\text{3 is odd}}_{Q}$. (P and Q comment have nothing to do with each other)

   T ⇒ T – True.

   example of a Theorem (needs proof) → (b) If $\underbrace{a \text{ is even}}_{P}$, then a^2 is even. Needs proof (will do later)

   open sentence ← depends on a, whether T/F

   (c) $\underbrace{\text{5 is even}}_{P}$, therefore $\underbrace{\text{3 is odd}}_{Q}$. – they have nothing to do with each other ↑ (but this is not relevant)

   False ⇒ True — True

   (d) 5 is even implies that 25 is even.

   True (again: F ⇒ F)
   (it's an example of (b), except here P is false). $a=5$

   (e) If a number a is even, then the number $2a + 3$ is odd.

   (f) For any integer a, the number $24a + 3$ is odd.

2. Find the set of all positive divisors of 60.

3. Prove that for any integer n, the number $n(n + 1)(n + 2)$ is divisible by 6.

convention: We imply that the statement should be true for all a if we say it is True.

About theorems:

Some of them (the majority) are stated in the form $P \Rightarrow Q$

Often, P and Q are open sentences, depending on some variables.

When we prove the theorem, we only argue about the variables that make P True.

(Also, to prove a theorem, we need to prove it for all values of the variables for which P is True).

In our example (b)

Theorem: If $\overbrace{a \text{ is even}}^{P(a)}$, then $\overbrace{a^2 \text{ is even}}^{Q(a)}$.

This means: need to prove it for all a.
sufficient to only consider even a
(a is odd $\Rightarrow$ anything)
makes P false

First, need
Definition: what is an even number?

An even number is a number of the form
$a = 2k$, where $k \in \mathbb{Z}$.
(k is an integer)

An odd number is a number of the form
$a = 2k+1$ for some $k \in \mathbb{Z}$. ← see below.

Even/odd applies only to integers.

$\mathbb{Z} = (\text{even}) \cup (\text{odd})$ ← union of sets

$\{n \in \mathbb{Z} : n = 2k \text{ for some } k \in \mathbb{Z}\} \cup \{n \in \mathbb{Z} : n = 2k+1 \text{ for some } k \in \mathbb{Z}\}$.

What does $a = 2k+1$ for some $k \in \mathbb{Z}$ mean?

A few examples: $a = 10$

Can you find $k \in \mathbb{Z}$ such that $a = 2k$?

yes: $k = 5$

Can you find $l \in \mathbb{Z}$ such that $a = 2l+1$?

cannot find such l

$$\boxed{\begin{array}{ll} l < 4, & 2l+1 < 10 \\ 10 = 2 \cdot 4 + 1 & - \text{False} \\ 10 = 2 \cdot 5 + 1 & - \text{False} \\ l > 5, & 2l+1 > 10 \end{array}}$$

Note: the fact l does not exist required proof

For any a, only one of these questions has the answer "yes".

If the first question has "yes" answer, then a is even

if the second one has "yes", then a is odd.

Question: where do we name our number?

If I talk about a number, have to give it a name: a, or n, or k, ...

"A number n is even if there is an integer k ... such that $n = 2k$."

new name for a new number.

Proof of our Theorem:

Thm: If $\overbrace{a \text{ is even}}^{P(a)}$ then a^2 is even.

Let a be an even integer. ← make $P(a)$ True

Then there is $k \in \mathbb{Z}$ such that $a = 2k$ — Interpret it using definitions

Then $a^2 = (2k)^2 = 4k^2$
$= 2 \cdot (2k^2)$

Then $a^2 = 2\ell$ where $\ell = 2k^2$ – an integer! as required to prove ■, Q.E.D.

symbols for end of proof

How to get there: Last line: want to have "Then a^2 is even"

this means, I want to show that there is an integer ℓ such that $a^2 = 2 \cdot \ell$

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   (b) If a is even, then a^2 is even.

   (c) 5 is even, therefore 3 is odd.

   (d) 5 is even implies that 25 is even.

   (e) If a number a is even, then the number $2a + 3$ is odd.

Let a be even. Then $a = 2k$ for some $k \in \mathbb{Z}$.

Then $2a+3 = 2(2k)+3 = 4k+3 = 2\cdot\underbrace{(2k+1)}_{= \ell}+1$

ℓ - integer.

Then $4k+3$ is odd so $2a+3$ is odd. QED.

   (f) For any integer a, the number $24a + 3$ is odd.

Pf: Let a be an integer.

Then $24a+3 = 2(12a+1)+1$. since $12a+1$ is an integer, $24a+3$ is odd.

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3. Prove that for any integer n, the number $n(n + 1)(n + 2)$ is divisible by 6.

Want: $2a+3$ is odd, so want to express $2a+3$ as $2\cdot\ell+1$

Complaint: Theorem is True, but its assumption was unnecessary! $2a+3$ is odd for any integer a, not just even a.

Our (e) is a special case of this general statement.

OK to prove a more general statement.

Divisibility

Def: Let a, b be integers, $b \neq 0$.
We say $b \mid a$ ("b divides a") if
$a = bk$ for some integer k.
(b is called a divisor of a)

see discussion in the book on 'Definitions'.

Hint for #3: from worksheet: (~~...~~ will prove next time Think about it!)

Lemma: $n \in \mathbb{Z}$. If $2 \mid n$ and $3 \mid n$, then $6 \mid n$.

a theorem used in a proof of a bigger theorem.

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2. Find the set of all positive divisors of 60.

$\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\} = \{n \in \mathbb{Z}_{n>0} : n \mid 60\}$

3. Prove that for any integer n, the number $n(n + 1)(n + 2)$ is divisible by 6.

<u>Proof in 3 steps</u> :

<u>Step 1</u>: Prove it is divisible by 2

<u>Step 2</u>: Prove it is divisible by 3

<u>Step 3</u>: Prove the <u>Lemma</u> (see prev. page)