Worksheet 1: Sets and Logic

1. Which of the following sentences are statements in the mathematical sense? For the ones that are statements, can you decide whether they are True or False?

(a) It is sunny outside right now.

Statement: True.

(b) Tomorrow the weather will be nice.

Statement: False. (Non-statement: can be made into a statement if you define 'nice').

(c) The 100th digit of the decimal expansion of \( \pi \) is 7.

Statement: We don't know T/F, but it has a False value.

(d) The digits of \( \pi \) encode the meaning of the Universe.

Non-statement.

(e) This statement is False.

Statement: Not ambiguous.

(f) This statement is True.

True statement? True works. Also a non-statement.

(g) For any consistent system of axioms there exists a statement about natural numbers that is true, but unprovable from these axioms.

Gödel's incompleteness theorem (statement once you define all ingredients, True).

(h) For some prime numbers \( p \), the number \( p + 2 \) is also prime.

True statement. Proof: take \( p = 3 \), \( p + 2 = 5 \).

(i) For all prime numbers \( p \), the number \( p + 2 \) is also prime.

False. Take \( p = 7 \), \( p + 2 = 9 = 3 \cdot 3 \) not prime.

(j) There exist infinitely many primes \( p \) such that the number \( p + 2 \) is also prime.

Twin prime conjecture: We do not know if it is True or False.
Discussion for the worksheet answers:

Recall: Statement: sentence that has

"truth value": True or False.

(not 'sometimes', or 'maybe').

(sometimes we don't know True or False,
but we know it is one or the other —
e.g. (c) in the worksheet).

Note: Objection to 12-14 on p.42 in the book:

they are asking to use formal logic to express

"All happy families are alike, all unhappy families
are different" — more or less a statement.

"human beings want to be good, but not too good,
not all the time" — more or less a statement

"A man should look for what it is, not
what he thinks should be" — we understand
what it means

? But it is a non-statement, I think.

I agree with it.
Discussion of worksheet, continued.

(c) "This statement is False?"

- If true, then we believe it, then it is false.
- If false, then the opposite: it is True.

Contradiction both ways.
So it is not a statement because it cannot be True or False.

We will learn Proof by Contradiction.
Many proofs eventually boil down to a construction of this sort, to get the contradiction.
1) There are infinitely many prime numbers.
   True (will 'prove' in the course)

   Now let's consider statement (i)
   we have: example: $p = 29 \quad p+2 = 31$ - works
   we don't see examples that are larger. So it seems the conjecture might be false?
   But can we prove it?

   Why false: as numbers get larger, there are more factors, try to see how many numbers can be prime at all, unlikely to get infinitely many pairs so close

   This is TWIN PRIME conjecture. - No-one can prove (yet) whether it is true or false.

   Upshot: one way of proving things about primes is to check every one of them - but only works for finitely many.
   - so we can make arguments proving things about infinitely many objects.

   Read about twin prime conjecture.
2. Are the following sets empty or not? When not empty, draw the set.

(a) The set of all \( x \in \mathbb{R} \) such that \( x^2 > 4 \) and \( x < 0 \).

(b) The set of all \( x \in \mathbb{R} \) such that \( x^2 > 4 \) and \( |x| < 2 \).

\[ \emptyset \text{ - empty set.} \]

(c) The set of all \( x \in \mathbb{R} \) such that \( x^2 \geq 4 \) and \( |x| \leq 2 \).

\[ S = \{ 2, -2 \} \]

(d) The set of all \( (x, y) \in \mathbb{R}^2 \) such that \( x^2 + y^2 = 1 \) and \( x < 0 \).

\[ \{ (x, y) \in \mathbb{R}^2 : (x^2 + y^2 = 1) \land (x < 0) \} \]

(e) \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = -1\} \).

- this set is empty.

because \( x^2 \geq 0, \ y^2 \geq 0, \) so \( x^2 y^2 \geq 0, \) so \( x^2 + y^2 \neq -1 \).

(f) \( \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = -1\} \).

hyperbola
Note about notation: I used slightly different terminology in class compared to our book.

Class:
- $\neg P$ (negation)

$P$ True $\iff \neg \neg P$ False
$\neg P$ True

Predicate:
- $P(x, y)$: statement whose truth depends on $x, y$

Example: $x^2 + y^2 = 1$

Open sentence:
- (what I called "predicate"
- the book calls "open sentence")

Notation:
- $A$: set
- $|A|$ = cardinality of the set $A$
- $|A|$ = number of its elements
- If $A$ is finite
- and "infinite" if not.

We will use $a \mid b$ "$a$ divides $b$"
if there is $k \in \mathbb{Z}$ such that $b = ka$.

Example: $3 \mid 36$ - True
($k = 12$)

$2 \mid 41$ (there is no such $k$. Integer)

That $41 = 2k$
Our next thing: Conditional statements

(implication)

Read 2.3

$p, q$ - statements.

We defined: $p \land q$, $p \lor q$, $\neg p = \neg p$

Now: Conditional statement: $p \Rightarrow q$

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"$p$ implies $q$". Will discuss this next class.