Worksheet 14: Injective and surjective functions; composition.

1. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is injective but not surjective. Can you make such a function from a finite set to itself?

2. Prove that the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(a, b) = 3a + 7b$ is surjective. Is this function injective?

3. Prove that among any six distinct integers, there are two whose difference is divisible by 5.

4. Let $f : \{1, 2, 3\} \to \{a, b, c, d\}$ be defined by $f(1) = a$, $f(2) = c$, $f(3) = d$. Let $g : \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$ be defined by $g(a) = 2$, $g(b) = 1$, $g(c) = 4$, $g(d) = 5$. Find the composition $g \circ f$.

5. Prove that if $f : A \to B$ is injective and $g : B \to C$ is injective, then $g \circ f : A \to C$ is injective. Is the converse statement true?
Today: Continue Functions

\[ f: A \rightarrow B \]

\[ \text{Def: } f: A \rightarrow B \text{ is called injective } \]
\[ f(a_1) \neq f(a_2) \text{ for any } a_1, a_2 \in A \text{ s.t. } a_1 \neq a_2. \]

(Also called one-to-one sometimes)

\[ \text{Def: } f: A \rightarrow B \text{ is called surjective (also called onto)} \]
\[ \text{If } \forall b \in B \exists a \in A \text{ s.t. } f(a) = b \]

(equivalently, \( \text{Range}(f) = B \))

\[ \text{Def: } f: A \rightarrow B \text{ is called bijective (one-to-one is often reserved for this)} \]
\[ \text{If it is both injective and surjective} \]

Examples:

1. \( f: \mathbb{R} \rightarrow \mathbb{R} \)
   \[ f(x) = x^2 \]
   Not injective because
   \[ x^2 = (-x)^2 \text{ for all } x \]
   (In fact, enough just to say: \( 1^2 = (-1)^2 \))

Not surjective (\( x^2 \geq 0 \) for all \( x \), so for example, \(-1 \notin \text{Range}(f) \))

Then \( \text{Range}(f) \neq \mathbb{R} \)

2. \( f: \mathbb{R} \rightarrow \mathbb{R} \)
   \[ f(x) = \frac{1}{x} \]
   Not actually a function from \( \mathbb{R} \) to \( \mathbb{R} \)!
   Not defined at 0!
   So we have \( f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \)

It is injective: \( \text{pf: contrapositive:} \)

Suppose \( f(x_1) = f(x_2) \)

Then \( \frac{1}{x_1} = \frac{1}{x_2} \) \( \text{(and } x_1, x_2 \in \mathbb{R} \setminus \{0\} \) \)
Then \( x_1 = x_2 \). This proves \( f \) is injective.

(3) \( f : \mathbb{R} \to \mathbb{R} \) defined by: \( f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ a, & x = 0 \end{cases} \)

If we want \( f \) to be injective,
then \( a \) has to be zero!
If we take \( a = 0 \), then \( f \) becomes bijective.

(4) \( f(x) = e^x \)

Strictly increasing \( \Rightarrow \) injective

(Direct proof: Let \( x_1 \neq x_2 \). Want to prove: \( f(x_1) \neq f(x_2) \).

Given: \( f \) is strictly increasing.

WLOG, \( x_1 < x_2 \). Then since \( f \) is
strictly increasing, \( f(x_1) < f(x_2) \). So \( f(x_1) \neq f(x_2) \).

\( f(x) > 0 \forall x \in \mathbb{R}, \) so Range \( f \) is \( (0, +\infty) \neq \mathbb{R} \).

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Worksheet question 2
To prove that \( f \) is surjective.
we need to show: \( \forall c \in \mathbb{Z}, \exists a, b \text{ s.t. } 3a + 7b = c \)

Many ways to prove it:

Way 1: first, take \( c = 1 \). If \( a = -2, b = 1 \),
we get: \( 3a + 7b = -6 + 7 = 1 \).

Then for any \( c \in \mathbb{Z} \), take \( a = -2c, b = c \)
and we get \( c, 1 = c \cdot (3 \cdot (-2) + 7, 1) = 3a + 7b \).

Way 2: Bezout's identity:
\[ 1 = \gcd(3, 7) \] — greatest common divisor.
we proved: \( \exists x, y \text{ s.t. } 1 = 3x + 7y \).
\( \frac{1}{2} \) (in fact, \( x = -2, y = 1 \) works.)

No, proceed as in Way 1.
Way 3: By cases: \[ c \equiv 0 \]
\[ c \equiv 1 \pmod{3} \]
\[ c \equiv 2 \]

1) if \( c \equiv 0 \pmod{3} \), then \( c = 3k \), take \( a = k, b = 0 \) done.

2) if \( c \equiv 1 \pmod{3} \), then: 1) represent 1:
\[ 1 = 3 \cdot (-2) + 7 \]

Then \( c = 3k + 1 = 3(k + 1) + 7 - 2 \) - done

3) if \( c \equiv 2 \pmod{3} \), same thing, represent 2:
\[ 2 = 3 \cdot (4) + 7 - 2 \]
Proceed as in case 2.

How to come up with this: \[ c = 3a + 7b \] — want.

So: \( c - 7b \) needs to be divisible by 3.

Now it's natural to consider cases \( c \pmod{3} \).

Back to question 1: let \( A \) be a finite set.

Can there be a function \( f: A \rightarrow A \) that is injective but not surjective?

\[ \rightarrow \quad \text{No!} = \text{Pigeonhole principle} \]

\[ \rightarrow \]

\[ A 
\[ \rightarrow \]

\[ A \]
Question 3: "rabbits": if you try to put n+1 or more rabbits into n cages, then at least two will wind up in the same cage.

"rabbits": the 6 given integers
"cages": congruence classes mod 5:
\([0], [1], [2], [3], [4] \subset 5 \) of them.

Then of my 6 integers, two have to end up in the same congruence class.

Composition of functions

Let \( f: A \rightarrow B \) - functions.
\( g: B \rightarrow C \)

Then their composition denoted by \( g \circ f \), is

\( g \circ f: A \rightarrow C \)
defined by: \( g(f(x)) \), for \( x \in A \).

\( g \circ f(x) \)

\( \circ \) in \( \LaTeX \)

Question 4 from Worksheet

A = \{1, 2, 3\}

\( g \circ f: A \rightarrow C \)
\( g \circ f: \{1, 2, 3\} \rightarrow \{1, 2, 3, 4, 5\} \)

\( g \circ f = \{(1, 2), (2, 4), (3, 5)\} \)

\( A \times C \)