Reminder: an equivalence relation on A partitions the set A into equivalence classes.

Example: $[n]_d$ = class of an integer modulo d

$= \{a \in \mathbb{Z} \mid a \equiv n \mod d\}$.

Key point (see 11.5): if you take two integers from the same class mod d (a, a')

and b, b' from some other class

then aa' and bb' end up in the same class.

And $a+a'$, $b+b'$ will be in the same class.

So: you can do arithmetic operations on congruence classes!

Ex: $[3^k]_4 = \underbrace{[3]_4^k} = [-1]_4^k = [(-1)^k] = \begin{cases} [1] & k \text{ even} \\ [-1] & k \text{ odd} \end{cases}$

because $3 \equiv -1 \mod 4$

Worksheet 13: Congruence of integers; Functions

1. Let $d \in \mathbb{N}$. Prove that

$$(a \equiv b \mod d) \wedge (a' \equiv b' \mod d) \Rightarrow aa' \equiv bb' \mod d.$$

$a \equiv b \mod d$ means: $d \mid b-a$, so $b-a = dk$ for some $k \in \mathbb{Z}$.

Similarly, $b' - a' = dl$ for some $l \in \mathbb{Z}$.

Then: $b = dk + a$ & $b' = dl + a'$: $bb' = (dk+a)(dl+a')$
$= d(k + l + dkl) + aa'$. Thus $d \mid bb' - aa'$.

2. Prove that if an integer a is written with the digits $a_n, \ldots, a_0$, then a and $a_0 + \cdots + a_n$ are in the same congruence class $\mod 9$.

example: $[123456]_9 = [1+2+3+4+5+6]_9$
$= [3]_9$.

3. Prove that for any integers a and b, the sum $a^2 + b^2$ lies in one of the classes $[1]$, $[0]$, or $[2]$ $\mod 4$. Deduce that the number 1000535 cannot be represented as a sum of two squares.

4. Prove that there do not exist integers a, b and c such that

$$12345678910111213 = a^2 + 25b^2 + 5c^2.$$

Textbook solution to #1

Want to prove: $d \mid bb' - aa'$

Write $bb' - aa' = b(b'-a') + \underline{ba' - aa'}$

$\quad\quad\quad\quad\quad\quad\quad\quad bb' - ba'$

$= \underbrace{b(b'-a') + a'(b-a)}_{\text{both are divisible by } d}$, so $d \mid bb' - aa'$.

by properties of congruences proved earlier.

Also prove: $a \equiv a' \mod d$
$b \equiv b' \mod d$

Then $a + b \equiv a' + b' \mod d$.

Consequence: we can do operations $(+, \times)$ on congruence classes mod d.

Can write $[a]\cdot[b] = [ab]$
(here $[\ ]$ is class mod d).
$[a] + [b] = [a+b]$

and these operations are well defined.

Question 2 explanation :

What I am saying is:

take a number, for example, 372

suppose we want to find its remainder mod 9. A quick way : add up its digits:

$\underline{3+7}+2 \equiv 1+2 \mod 9$

Answer : 3.

Our problem says : $\overline{abc}$ = number written with digits a,b,c

$$\overline{abc} \equiv a+b+c \mod 9 \quad (\text{or } \mod 3)$$

$$\overline{abc} = 100a+10b+c$$

Want to prove: $\overline{abc} \equiv a+b+c \mod 9$

$\iff (100a+10b+c)-(a+b+c)$ is divisible by 9.

we get: $99a+9b$ – it is divisible by 9. and we are done.

In general: Lemma $\forall n \in \mathbb{N}$, $10^n \equiv 1 \mod 9$

Pf of Lemma: $[10^n]_9 = [10]_9^n = [1]_9^n = [1]$.

(this is saying: $10 \equiv 1 \mod 9$

$\Rightarrow 10^n \equiv 1^n \mod 9$ by properties of congruences).

Now, suppose we have a number :

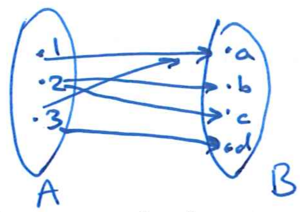
$A = \overline{a_n a_{n-1} --- a_0}$ written with the digits $a_0, a_1, \ldots, a_n$

Then $A = a_n \cdot 10^n + a_{n-1} 10^{n-1} + \ldots + a_0$

$\equiv a_n \cdot 1 + a_{n-1} \cdot 1 + \ldots + a_0 \mod 9$ by Lemma.

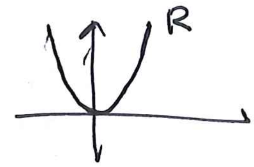
So we are done!

5. Let $A = \{1, 2, 3\}$, and let $B = \{a, b, c, d\}$. Let $R = \{(1, a), (2, b), (2, c), (3, a), (3, d)\}$ - a relation from A to B. Draw a diagram representing this relation.



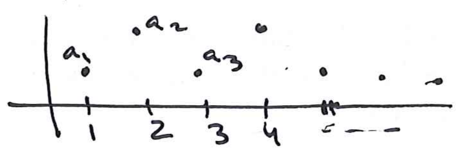
6. Represent the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ as a relation.

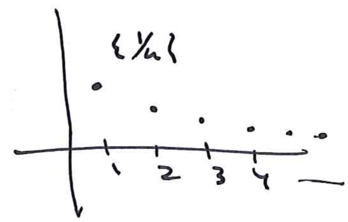
$R = \{(x, x^2) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$



7. Represent the sequence $a_n = 1/n$ as a relation; think of it as a function from $\mathbb{N}$ to $\mathbb{R}$.

A sequence is a function: $\mathbb{N} \to \mathbb{R}$





~~8. Give an example of a function that is injective but not surjective.~~

As a relation our sequence is:

$\{(n, \frac{1}{n}) \mid n \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{R}$.

It is a function from $\mathbb{N}$ to $\mathbb{R}$

domain codomain.

Functions (Chapter 12!) (Read 12.1 – 12.3)

function ⇝ graph ⊂ (domain) × (codomain)

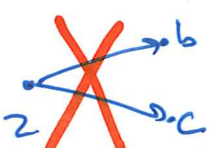
it is a relation!

$$x \longrightarrow f(x)$$

Def: A function $f: A \to B$ is a relation R on $A \times B$ (R from A to B), such that every element of A appears exactly once as the first coordinate of an element of R:

$$\left(\forall a \in A, \ \underset{\substack{\uparrow \\ \text{exists} \\ \text{unique}}}{\exists!} \ (x,y) \in R \text{ such that } x = a \right)$$

Example: our question 5 in the worksheet is a relation from A to B which is NOT a function: because (2,b) and (2,c) both are in R

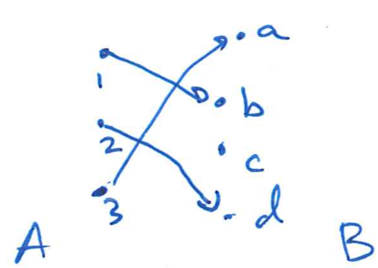


not allowed for a function.

Example: $A = \{1,2,3\}$

$B = \{a,b,c,d\}$

$R = \{(1,b),(2,d),(3,a)\}$



Every element of A has exactly one arrow coming out of it!

Def In this situation, $f: A \to B$
A is called the <u>domain</u> of f
and B – the <u>codomain</u> of f.

<u>Remarks</u>: The way we defined it here, f is defined at <u>every element of the domain</u>

(in calculus before, you have $f: \mathbb{R} \to \mathbb{R}$
but it's maybe not defined at some points,
e.g. $1/x$ not def'd at $x=0$.

Here we write: $f(x) = \frac{1}{x}$

$$f: \underline{\underline{\mathbb{R} \setminus \{0\}}} \to \mathbb{R}$$

Codomain <u>contains</u> range of f but does not have to equal it:

$f(x) = x^2$ $f: \mathbb{R} \to \mathbb{R}$ – OK.

range$(f) = \{y \in \mathbb{R} \mid y \geq 0\}$.