Worksheet 12: Cartesian product, indexed collections, relations

1. Let $A_n = [0, \frac{1}{n}] \times [0,n]$. Draw the picture representing $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$. (Both should be subsets of $\mathbb{R}^2$).

$[0, \frac{1}{n}]$ means the interval from 0 to $\frac{1}{n}$.

[end points included]

2. Let $R$ be the relation $a \equiv b \mod 3$ on the set $A = \{0, 1, 2, 3, 4, 5\}$. Write this relation as a subset of $A \times A$.

3. Let $A = \{1, 2, 3\}$, and let $B = \{a, b, c, d\}$. Let $R = \{(1, a), (2, b), (2, c), (3, a), (3, d)\}$ - a relation from $A$ to $B$. Draw a diagram representing this relation.

4. Let $A = \{x, y, z, w\}$ and let $R = \{(x, x), (x, y), (y, z), (w, w), (y, x), (z, y)\}$. Is this relation symmetric? Is it reflexive? Is it transitive?

5. Let $A = \{a, b, c, d, e, f\}$ and let $R_1 = \{(a, a), (a, b), (b, e), (c, c), (d, e)\} \subset A \times A$. Find $R_2$ - the smallest relation containing $R_1$ that is an equivalence relation. Then find the partition of $A$ into equivalence classes according to $R_2$. 
Let us draw $A_1 = [0,1] \times [0,1]$ 
$= \{(x,y) \mid x \in [0,1], y \in [0,1]\}$

Reminder: $\bigcup_{n \in \mathbb{N}} A_n = \text{the union of all these sets}$
$= \{ (x,y) \mid \exists n: (x,y) \in A_n \}$
$= \{ (x,y) : \exists \, n : (0 \leq x \leq \frac{1}{n}) \land (0 \leq y \leq n) \}$.

$\bigcup_{n \in \mathbb{N}} A_n = \text{illustration for the union of all } A_n$
The intersection: \( \cap \{ A_n \}_{n \in \mathbb{N}} \)

\[ \begin{align*}
\cap \{ A_n \}_{n \in \mathbb{N}} & \triangleright \{(x,y) \mid (x,y) \in A_n \text{ for all } n \}\,\forall n, (x,y) \in A_n \\
& = \{(x,y) \mid 0 \leq x \leq \frac{1}{n} \text{ for all } n \in \mathbb{N} \}
& \quad \text{and } 0 \leq y \leq 1 \text{ for all } n \in \mathbb{N} \}
& = \{(x,y) \mid x=0 \land 0 \leq y \leq 1 \} = \{(0) \times [0,1]\}
\end{align*} \]

\[ \text{interval (at thickness!)} \]

\[ \text{Note: If we define } B_n = [0, \frac{1}{n}] \times [0,1], \]
\[ \text{then } \cap \{ B_n \}_{n \in \mathbb{N}} = \emptyset \]

\[ \text{Proof: } \begin{align*}
\cap \{ B_n \}_{n \in \mathbb{N}} & = \{(x,y) \mid 0 \leq x \leq \frac{1}{n} \land 0 \leq y \leq 1 \text{ for all } n \in \mathbb{N} \}
& \text{for all } n \in \mathbb{N} \}
& \text{for all } n \in \mathbb{N} \}
\end{align*} \]

Since for every \( x > 0 \), exists \( n \in \mathbb{N} \) such that \( \frac{1}{n} < x \), we have:

\[ \{ x : 0 \leq x \leq \frac{1}{n} \text{ for all } n \in \mathbb{N} \} = \emptyset. \]

Then \( \cap \{ B_n \}_{n \in \mathbb{N}} = \emptyset \) because the set of the first coordinates \( x \) is empty!
Simpler question: Let $B_r = (0, \frac{1}{r})$ for $r \in \mathbb{R}^+$. What is $\bigcap_{r \in \mathbb{R}^+} B_r$?

Claim: $\bigcap_{r \in \mathbb{R}^+} B_r = \emptyset$.

Proof: Want to prove: there does not exist $x$ that belongs to all sets $B_r$ at once.

$$\bigcap_{r \in \mathbb{R}^+} B_r = \{ x \mid \forall r \in \mathbb{R}, x \in B_r \}$$

$$= \{ x \mid \forall r \in \mathbb{R}, 0 < x < \frac{1}{r} \}$$

Now want to prove that this set is empty. Suppose it was not empty. Let $x_0$ be some element of this set. Then $x_0$ satisfies:

$$0 < x_0 < \frac{1}{r} \quad \text{for every } r \in \mathbb{R}^+$$

This is false: take $r = \frac{1}{x_0}$, then $x_0 < \frac{1}{r}$ is false.

/Note: this is equivalent to saying: \[ \lim_{r \to 0} \frac{1}{r} = 0 \]

Then $x_0$ doesn't exist, so the intersection is empty.
New question: Let \( A_r = [0, r] \times [0, \frac{1}{r}] \) for \( r \in \mathbb{R}_+ \) - set of positive real numbers.

What is \( \bigcup_{r \in \mathbb{R}_+} A_r \) and \( \bigcap_{r \in \mathbb{R}_+} A_r \)?

\[ \bigcap_{r \in \mathbb{R}_+} A_r = \{(0,0)\} \]

pf: homework.
Relations:

Examples we know:

1. Relation of being \( \geq \) among real natural numbers

   write \( x \geq y \) \( \iff \) \( x \) and \( y \) are in the relation "\( \geq \)" to each other.

2. \( x = y \): the equality.

Formal way to think about these:

Def: A relation \( R \) on a set \( A \) is a subset of \( A \times A \).

We say \( x \) is in relation \( R \) to \( y \) (notation: \( xRy \)) if \( (x, y) \in R \).

Write our familiar examples in this way:

\[
\begin{align*}
&x = y \\
&\text{let } A = \mathbb{R} \\
&\text{Relation of equality as a subset } E \subseteq \mathbb{R} \times \mathbb{R} ^{\text{"equality"}}
\end{align*}
\]

\( xEy \) means \( x = y \)
Question 2: \( a \equiv b \mod 3 \)

\[ \text{def} = \{ (a, b) \in A \times A \mid 3 \mid a - b \} \]

\[ \text{refl: } 3 \text{ divides } a - b \]

\[ \{ (0,0), (1,1), (2,2), \ldots, (5,5) \} \]

\[ (0,3), (3,0) \] symmetric

\[ (1,4), (4,1) \] symmetric

\[ (2,5), (5,2) \] symmetric

Def: A relation \( R \subseteq A \times A \)

reflexive if \( \forall x \in A, (x, x) \in R \)

symmetric if \( \forall x, y \in A, (x, y) \in R \Rightarrow (y, x) \in R \)

transitive if \( x Ry \) and \( y Rz \) then \( x Rz \),

i.e., \( (x, y) \in R \land (y, z) \in R \Rightarrow (x, z) \in R \).

A relation satisfying these is called a \underline{equivalence relation}.

Examples: equality \underline{- equivalence relation}

\( x \geq y \) \underline{- Not symmetric (reflexive and transitive)}

\( \not \text{not an equiv. relation.} \)

\( \equiv \mod d \) \underline{- congruence \mod d \ - \ an \ equiv. relation.}

(prove it!)

Questions 3-5 will be discussed on Thursday.