Worksheet 12: Cartesian product, indexed collections, relations

1. Let $A_n = [0, \frac{1}{n}] \times [0, n]$. Draw the picture representing $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$. (Both should be subsets of $\mathbb{R}^2$).

$[0, \frac{1}{n}]$ means the interval from 0 to $\frac{1}{n}$:

(end points included)

2. Let R be the relation $a \equiv b \mod 3$ on the set $A = \{0, 1, 2, 3, 4, 5\}$. Write this relation as a subset of $A \times A$.

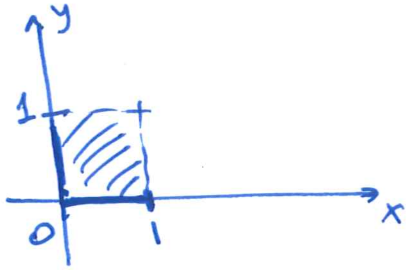
3. Let $A = \{1, 2, 3\}$, and let $B = \{a, b, c, d\}$. Let $R = \{(1, a), (2, b), (2, c), (3, a), (3, d)\}$ - a relation from A to B. Draw a diagram representing this relation.

4. Let $A = \{x, y, z, w\}$ and let $R = \{(x, x), (x, y), (y, z), (w, w), (y, x), (z, y)\}$. Is this relation symmetric? Is it reflexive? is it transitive?

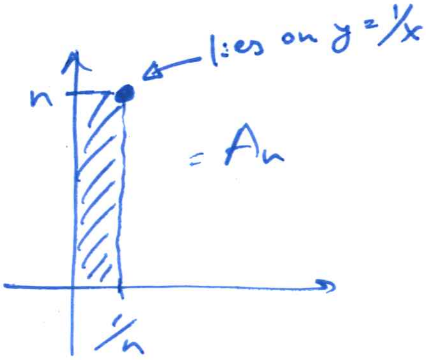
5. Let $A = \{a, b, c, d, e, f\}$ and let $R_1 = \{(a, a), (a, b), (b, e), (c, c), (d, e)\} \subset A \times A$. Find R_2 - the smallest relation containing R_1 that is an equivalence relation. Then find the partition of A into equivalence classes according to R_2.

Let us draw $A_1 = [0,1] \times [0,1]$

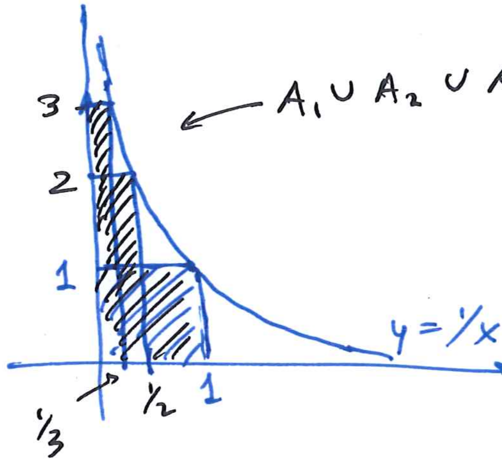
cartesian product



$= \{ (x,y) \mid x \in [0,1],\ y \in [0,1] \}$



$= A_n$



Reminder: $\bigcup_{n \in \mathbb{N}} A_n$ = the union of all these sets

$\stackrel{def}{=} \{ (x,y) \mid \exists n: (x,y) \in A_n \}$

$= \{ (x,y) : \exists n: (0 \le x \le 1/n) \wedge (0 \le y \le n) \}$.

$\bigcup_{n \in \mathbb{N}} A_n =$

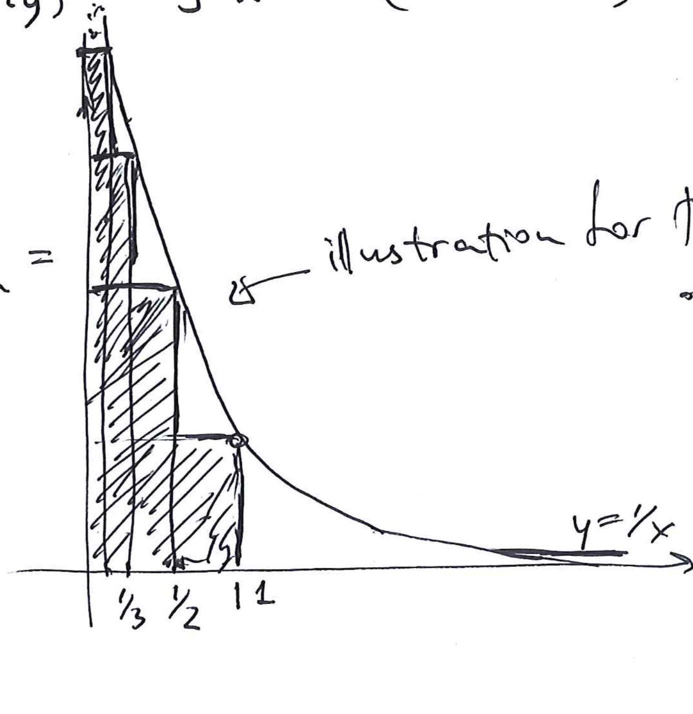


illustration for the union of all A_n

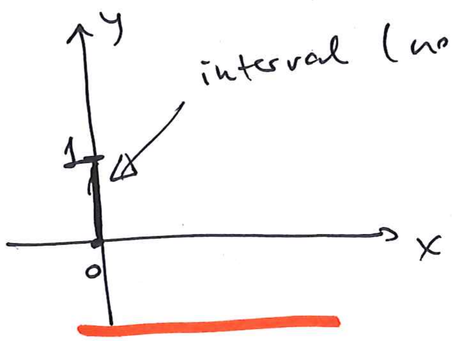
The intersection: $\bigcap_{n\in\mathbb{N}} A_n$

$$\stackrel{\text{def}}{=} \{(x,y) \mid (x,y)\in A_n \text{ for all } n\}$$

$\forall n,\ (x,y)\in A_n$

$$= \{(x,y) \mid 0\le x\le \tfrac{1}{n} \text{ for all } n\in\mathbb{N} \text{ and } 0\le y\le n \text{ for all } n\in\mathbb{N}\}$$

$$= \{(x,y) \mid x=0 \wedge 0\le y\le 1\} = \{0\}\times[0,1]$$



Note: If we define $B_n = (0,\frac{1}{n}]\times[0,n]$

difference from A_n: 0 not included.

Then $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$

Proof: $\bigcap_{n\in\mathbb{N}} B_n \underset{\text{def}}{=} \{(x,y) \mid 0<x\le\frac{1}{n} \wedge 0\le y\le n \text{ for all } n\in\mathbb{N}\}$

Since for every $x>0$, exists $n\in\mathbb{N}$ such that $\frac{1}{n}<x$, we have:

$$\{x : 0<x\le\tfrac{1}{n} \text{ for all } n\} = \emptyset.$$

Then $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$ because the set of the first coordinates x is empty!

Simpler question: Let $B_r = (0, \frac{1}{r})$ for $r \in \mathbb{R}_+$

What is $\bigcap_{r \in \mathbb{R}_+} B_r$?

Claim: $\bigcap_{r \in \mathbb{R}_+} B_r = \emptyset$.

Proof: Want to prove: there does not exist x that belongs to all sets B_r at once.

$$\bigcap_{r \in \mathbb{R}_+} B_r \underset{\text{def of intersection}}{=} \{ x \mid \forall r \in \mathbb{R}, \ x \in B_r \}$$

$$\underset{\text{def of } B_r}{=} \{ x \mid \forall r \in \mathbb{R}, \ 0 < x < \tfrac{1}{r} \}$$

Now want to prove that this set is empty.

Suppose it was not empty. Let x_0 be some element of this set. Then x_0 satisfies:

$$0 < x_0 < \frac{1}{r} \text{ for every } r \in \mathbb{R}_+$$

This is false: take $r > \frac{1}{x_0}$, then $x_0 < \frac{1}{r}$ is false!

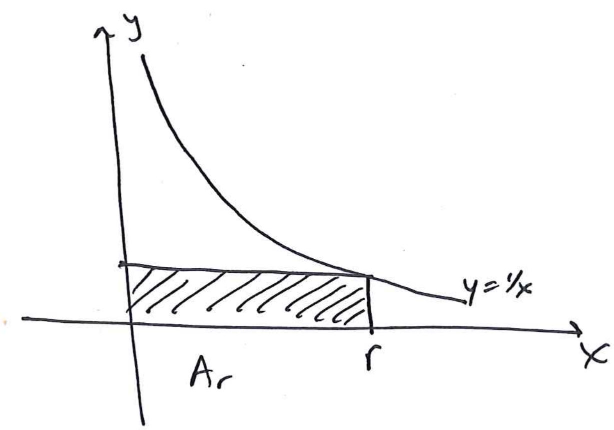
Then x_0 doesn't exist, so the intersection is empty.

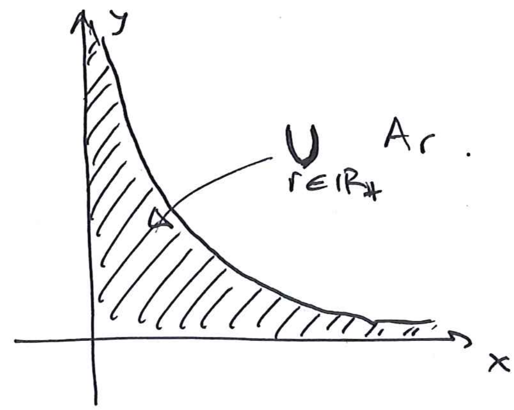
Note: this is equivalent to saying: $\lim_{r \to \infty} \frac{1}{r} = 0$

<u>New question</u> : let $A_r = [0, r] \times [0, \frac{1}{r}]$

for $r \in \mathbb{R}_+$ - set of positive real numbers.

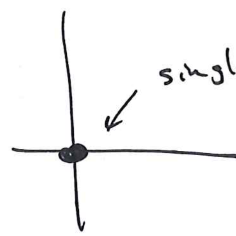
What is $\bigcup_{r \in \mathbb{R}_+} A_r$ and $\bigcap_{r \in \mathbb{R}_+} A_r$?





$$\bigcap_{r \in \mathbb{R}_+} A_r = \{(0,0)\}$$

pf : homework.



single point.

Relations:

examples we know

① Relation of being $\geq$ among real / natural / integers numbers

write $x \geq y$ ← x and y are in the relation "$\geq$" to each other.
$2 \geq 1$

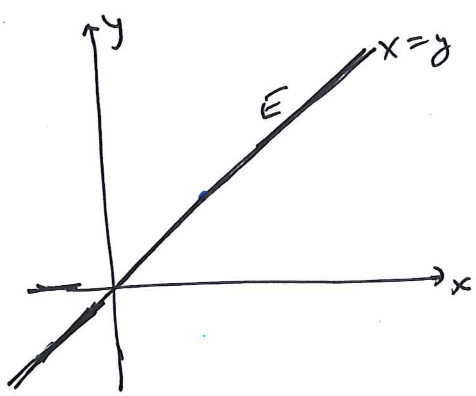
② $x = y$: the equality.

Formal way to think about these:

Def: A relation R on a set A is a subset of $A \times A$.

We say x is in relation R to y (notation: xRy) if $(x, y) \in R$.

Write our familiar examples in this way:



let $A = \mathbb{R}$
Relation of equality as a subset $E \subset \mathbb{R} \times \mathbb{R}$
"equality"

xEy means $x = y$

Question 2: ← from worksheet

$a \equiv b \mod 3$

$\stackrel{\text{def}}{=} \{ (a,b) \in A \times A \mid 3 \mid a-b \}$

3 divides A − B

$= \{ (0,0), (1,1), (2,2) \ldots, (5,5),$ ← reflexive

$(0,3), (3,0)$

$(1,4), (4,1)$ ← symmetric

$(2,5), (5,2) \}$

Def: A relation $R \subset A \times A$ is called

reflexive if $\forall x \in A$, $(x,x) \in R$

symmetric if $\forall x, y \in A$ $(x,y) \in R \Rightarrow (y,x) \in R$

transitive if xRy and yRz then xRz,

i.e., $(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$.

A relation satisfying these is called an equivalence relation.

Examples:
- equality − equivalence relation
- $x \geq y$ − Not symmetric (reflexive and transitive)

  not an equiv. relation.
- $\equiv \mod d$ − congruence mod d − an equiv. relation.

  (prove it!)

Questions 3−5 will be discussed on Thursday.