Thursday Feb 27

Today: sets, proofs of the statements about sets, Cartesian products.

(from the exam) — and from worksheet 11 ⇐ see next few pages.

Problem 2: $\{T_a\}_{a \in \mathbb{R}}$ — collection of sets indexed by reals.

The set T_a depends on a.

Somebody gives you $a \in \mathbb{R}$. Then you make the set $T_a = \{x \in \mathbb{R} : x \geq 0 \wedge x < a-2\}$

Need to prove: $T_a = \emptyset \iff a \in (-\infty, 2]$

$\iff a \leq 2$.

$\Leftarrow$ easy: if $a \leq 2$ then $x \geq 0 \wedge x < a-2$ are incompatible:

$x < a-2 \Rightarrow x < 2-2 = 0$

so get: $x \geq 0 \wedge x < 0$ — no such x exists.

$\Rightarrow$ **General rule**: Never use a new variable without writing a sentence:

Let x be ...

Let $a \in \mathbb{R}$. We make T_a.

Let x be an element of T_a. ← wait! what if $T_a = \emptyset$?

Hard to do anything with the empty set.

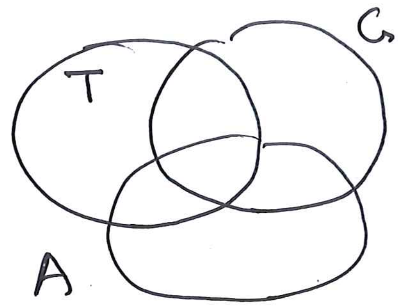
Suggest: we use contrapositive!

~~[illegible]~~. Suppose $a > 2$. Want to prove: $T_a \neq \emptyset$.

If $a > 2$ then $[0, a-2) \neq \emptyset$

(even better: take $x = 0$, it satisfies the conditions).

Aliens question (from Worksheet 10 from last class)



T = set of aliens who can watch TV

A = set of aliens who have antennae

G = set of aliens who are green

2) $G \cap \bar{A} \subseteq \bar{T}$
1) $T \not\subseteq A$

$\Big\}$ statement of our problem in terms of sets.

want to find whether $T \not\subseteq G$

True: existence proof: $\exists\ x \in T \wedge x \notin G$.

want to show:

To show this: given $T \not\subseteq A \Rightarrow$ exists $x_0 \in T$ $\wedge$ $x_0 \notin A$.

Since $x_0 \notin A$, by the statement (2) x_0 cannot be in G (otherwise it would not be in T)

So our $x_0 \in T$, and $x_0 \notin G$, so our claim is proved.

Instead of ~~sets~~ sets, could use open sentences:

$T(x)$ = x can watch TV $\qquad$ $A(x)$ - x has antennae

$G(x)$ - x is green.

Then our problem: $T(x) \not\Rightarrow A(x)$

$\Leftrightarrow \exists\, x_0 : \ T(x_0) \wedge (\sim A(x_0))$

we introduce a quantifier

need to introduce a quantifier, sometimes.

Similar issue on exam:

set of all integers who have a strict divisor > 7

$$= \{ n \in \mathbb{Z} : \exists\, b \in \mathbb{N} \text{ s.t. } b \mid n \wedge b \neq n \wedge b > 7 \}$$

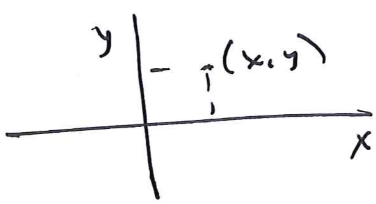
who is b?

Cartesian products:

Def: $A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$.

A, B - sets

Example $\mathbb{R} \times \mathbb{R} = \{ (x,y) \mid x, y \in \mathbb{R} \}$ - the plane



Note: Let $|A| = n$, $|B| = m$ ← two finite sets.

Then $|A \times B| = nm$ ← m choices for the second component.

n choices for the first component

Worksheet 11: Sets; indexed collections

1. Let A, B, C be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (\cancel{B}A \times C)$.

   See last page

2. Let $T_a = \{x \in \mathbb{R} : x \geq 0 \wedge x < a - 2\}$. Prove that if $T_a = \emptyset$ then $a \leq 2$.

   What is wrong with the following start of an argument:

   What is x ?

   > "Since $x \geq 0$ and $x < a - 2$ we must have $0 \leq x < a - 2$. Then ... "

• To prove: $T_a = \emptyset \Rightarrow \ldots$, better to start with something about T_a.

If $T_a = \emptyset$ (as we must assume) then who is x?

3. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be any function. Fon $N \in \mathbb{N}$, let $A_N = \{x \in \mathbb{R} : f(x) > N\}$.

   Prove that:

   (a) $\lim_{x \to +\infty} f(x) = +\infty$ if and only if for every N, there exists $m > 0$ such that $A_N \supseteq (m, +\infty)$.

   (b) Prove that $\cap_{N \in \mathbb{N}} A_N = \emptyset$.

   Will discuss next class

#1): Need to prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$\overset{\text{def}}{=} \{(a, x) \mid a \in A,\ x \in B \cap C\}$$

$$RHS = \{(a, b) \mid a \in A,\ b \in B\} \cap \{(a, c) \mid a \in A,\ c \in C\}$$

1) $LHS \subseteq RHS$:

$(a, x) \in A \times B \cap C \Rightarrow a \in A,\ x \in B \wedge x \in C$

Then $(a, x) \in A \times B \wedge (a, x) \in A \times C$

$\Rightarrow (a, x) \in (A \times B) \cap (A \times C)$.

2) $\Leftarrow$: $RHS \subseteq LHS$.

Let $(a, b) \in (A \times B) \cap (A \times C)$.

Then $a \in A$ and $b \in B$ and $b \in C$

so $(a, b) \in A \times (B \cap C)$.