Today: Sets, proofs of the statements about sets, Cartesian products.

(from the exam) and from worksheet 11 see next few pages.

Problem 2: \( \{ \text{Ta} \}_{a \in \mathbb{R}} \) — collection of sets

indexed by reals.

The set \( \text{Ta} \) depends on \( a \).

Somebody gives you \( a \in \mathbb{R} \). Then you make the set \( \text{Ta} = \{ x \in \mathbb{R} : x > 0 \land x < a - 2 \} \).

Need to prove: \( \text{Ta} = \emptyset \iff a \in (-\infty, 2] \iff a \leq 2 \).

\( \leq \) easy: if \( a \leq 2 \) then \( x > 0 \land x < a - 2 \) are incompatible:

\( x < a - 2 \implies x < 2 - 2 = 0 \)

so get: \( x > 0 \land x < 0 \) — no such \( x \) exists.

\( \implies \) General rule: Never use a variable without writing a sentence:

let \( x \) be...

let \( a \in \mathbb{R} \). We make \( \text{Ta} \).

let \( x \) be an element of \( \text{Ta} \). \( \leftarrow \) Wait! What if \( \text{Ta} = \emptyset \)?

Hard to do anything with the empty set.

Suggest: we use contrapositive!

Suppose \( a > 2 \). Want to prove:

\( \text{Ta} \neq \emptyset \).

If \( a > 2 \) then \( \{ 0, \frac{a}{2} - 2 \} \neq \emptyset \)

(even better: take \( x = 0 \), it satisfies the conditions).
Aliens question (from Worksheet 10 from last class)

\[ T = \text{set of aliens who can watch TV} \]
\[ A = \text{set of aliens who have antennae} \]
\[ G = \text{set of aliens who are green} \]

2) \( G \cap \overline{A} \subseteq T \) / statement of our problem in terms of sets.
1) \( T \not\subseteq A \)

Want to find whether \( T \not\subseteq G \)

True! existence proof: \( \exists x \in T \cap x \notin G \).

\[ \quad \text{want to show:} \]

To show this: given \( T \not\subseteq A \Rightarrow \exists x_0 \in T \land x_0 \notin A \).

Since \( x_0 \notin A \), by the statement (2)
\( x_0 \) cannot be in \( G \) (otherwise it would not be in \( T \))

So our \( x_0 \in T \), and \( x_0 \notin G \), so our claim is proved.

Instead of sets, could use open sentences:

\[ T(x) = \text{x can watch TV} \]
\[ A(x) = \text{x has antennae} \]
\[ G(x) = \text{x is green} \]

Then our problem: \( T(x) \not\subseteq A(x) \)

\[ \Leftrightarrow \exists x_0 : \quad T(x_0) \land \lnot(A(x_0)) \]

we introduce a quantifier
Similar issue on exam:

Set of all integers who have a strict divisor \( > 7 \)

\[ \{ n \in \mathbb{Z} : \exists b \in \mathbb{N} \text{ s.t. } b \mid n \land b \neq n \land b > 7 \} \]

\[ b \text{ who is } b? \]

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Cartesian products:

Def: \( A \times B = \{ (a,b) \mid a \in A, b \in B \} \).

\( A, B \)-sets

Example: \( \mathbb{R} \times \mathbb{R} = \{ (x,y) \mid x, y \in \mathbb{R} \} \) - the plane

\[ \begin{array}{c}
\text{y} \\
\downarrow \\
\text{f}(x,y) \\
\downarrow \\
\text{x}
\end{array} \]

Note: Let \( \lvert A \rvert = n \), \( \lvert B \rvert = m \) \( \leftarrow \) two finite sets.

Then \( \lvert A \times B \rvert = n \cdot m \leftarrow m \) choices for the second component.

\( \left. \begin{array}{l}
\begin{array}{c}
\lvert A \rvert \text{ choices} \\
\lvert B \rvert \text{ choices for the first component}
\end{array} \\
\end{array} \right\} \]

\( \left( \text{second component} \right) \)
Worksheet 11: Sets; indexed collections

1. Let $A, B, C$ be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

2. Let $T_a = \{x \in \mathbb{R} : x \geq 0 \land x < a - 2\}$. Prove that if $T_a = \emptyset$ then $a \leq 2$.

   What is wrong with the following start of an argument:

   \begin{quote}
   "Since $x \geq 0$ and $x < a - 2$ we must have $0 \leq x < a - 2$.
   Then ..."
   \end{quote}

   To prove: $T_a = \emptyset \implies \ldots \ldots$, better to start with something about $T_a$.

   If $T_a = \emptyset$ (as we must assume)

   then $x$?

3. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be any function. For $N \in \mathbb{N}$, let $A_N = \{x \in \mathbb{R} : f(x) > N\}$.

   Prove that:

   (a) $\lim_{x \to +\infty} f(x) = +\infty$ if and only if for every $N$, there exists $m > 0$ such that $A_N \supseteq (m, +\infty)$.

   (b) Prove that $\cap_{N \in \mathbb{N}} A_N = \emptyset$.

   \[ \text{Will discuss next class} \]
1) Need to prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Let $\{(a, x) \mid a \in A, x \in B \cap C\}$

$\text{RHS} = \{(a, b) \mid a \in A, b \in B \} \cap \{(a, c) \mid a \in A, c \in C\}$

1) LHS $\leq$ RHS:

$(a, x) \in A \times (B \cap C) \Rightarrow a \in A, x \in B \cap x \in C$

Then $(a, x) \in A \times B$ and $(a, x) \in A \times C$

$\Rightarrow (a, x) \in (A \times B) \cap (A \times C)$.

2) LHS $\leq$ RHS:

Let $(a, b) \in (A \times B) \cap (A \times C)$.

Then $a \in A$ and $b \in B$ and $b \in C$

So $(a, b) \in A \times (B \cap C)$. 