Worksheet 8: Proof or disproof?

Decide whether the following statements are True or False, and prove/disprove them:

1. Let \( \{u_n\} \) be a sequence of real numbers. \( \text{Reminder: we say that } u_n \to +\infty \text{ if } \forall N \exists m, (n \geq m \Rightarrow u_n > N). \)

(a) If \( u_n \to +\infty \), then \( -2u_n \to -\infty \).

\( \{u_n\} \) is a sequence s.t. \( u_n \to +\infty \)

TRUE. see next page (p. 2)

(b) If \( u_n \to +\infty \) then there does not exist \( m \) such that \( u_m < 0 \).

FALSE. Disproof: find a counterexample

see p. 3

(c) If \( u_n \to +\infty \) then \( \forall n \quad U_{n+1} \geq U_n \).

If \( u_n \to +\infty \), then \( \{u_n\} \) is increasing.

FALSE. see p. 4

(d) The converse of (c). (First state this converse).

If \( \forall u_n \), \( U_{n+1} \geq U_n \), then \( u_n \to +\infty \) - the converse.

FALSE. Counterexample: \( u_n = 1 + \frac{1}{n} \)

2. \( \exists x \), \( \forall y \quad x^2 - y < 0 \).

\( \exists x \), \( \forall y \quad x^2 - y < 0 \)

FALSE. see p. 6

Negation: \( \forall x \), \( \exists y \quad x^2 - y \geq 0 \).

3. \( \forall y \), \( \exists x \quad x^2 - y < 0 \).

FALSE. see p. 6
Ps of 1(a). Let \( u_n \to +\infty \).

Sequence: doesn't have to have a formula

\[ \begin{align*}
    u_1 &= 1, \text{ any number} \\
    u_2 &= \pi \\
    u_3 &= e^3 + 2 \\
    u_4 &= -7 \\
    u_5 &= 1001 \\
    \vdots
\end{align*} \]

Then, by definition,

\[ \forall N \exists m : n \geq m \Rightarrow u_n > N. \]

Want to prove: first need a definition of what it means that \( u_n \to -\infty \).

Make this definition:

\[ \forall N \exists m : n \geq m \Rightarrow u_n < -N \]

(to emphasize, could write:

\[ \forall N > 0 \exists m \text{ s.t. } n \geq m \Rightarrow u_n < -N \]

(The numbers \( u_n \) have to get "large negative" eventually)

Want to prove: Let \( v_n = -2u_n \).

\[ \forall N > 0 \exists m : n \geq m \Rightarrow v_n < -N \]

Let \( N \) be given.

we want: \( v_n < -N \), i.e. \( -2u_n < -N \)

\[ \Rightarrow u_n > \left[ \frac{N}{2} \right] \]

By definition, \( \exists m \) such that

\[ \forall n \geq m, \text{ we have } u_n > \left[ \frac{N}{2} \right] \]

Then for this \( m \), we'll have \( v_n < -N \), so we are done.
Want an example of a sequence \( \{u_n\} \) such that:

- \( u_n \to \infty \) and exists \( m \) s.t. \( u_m < 0 \).

Examples: \(-1, 0, 1, 2, 3, \ldots, n, \ldots\)

\[ u_n = n - 2 \]

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Comment: To disprove we could try to prove the negation is true:

"\( u_n \to \infty \) doesn't imply that there are no negative \( u_m \)"

there exists a sequence \( \{u_n\} \) s.t. \( u_n \to \infty \) and \( \exists m: u_m < 0 \).

why did we put this here? — because our statement was about all sequences even though we did not write that explicitly:
(c) "False because it just has to 'stay large','
doesn't have to increase.

Use our intuition to help us find a counterexample.

Example: \( U_n = n^3 - 5n^2 \)

To prove it works, we need to prove:
1) \( \lim_{n \to \infty} U_n = \infty \)
2) \( \exists n \) s.t. \( U_n < U_m \)

Given \( N \), you have to find \( m \) s.t.
\[ n \geq m, \quad n^3 - 5n^2 > N \]

OK counterexample but it gives us more work than we want.
Easier: suggestion: \( U_n = \begin{cases} -n & \text{if } n \text{ is even} \\ -3n & \text{if } n \text{ is odd} \end{cases} \)

Does it work? \(-1, 2, -3, 4, -5, 6, \ldots\)

Unfortunately doesn't work.

Need: \( \lim_{n \to \infty} U_n \) for all \( n \geq m \)

That we find:

Need it for all \( n \geq m \) not just even \( n \).

Suggestion: \( U_n = n^2 \). \( U_n \to \infty \) is easy to check

Unfortunately it is increasing so gives no counterexample.

Suggestion: \( U_n = N \) constant sequence only makes sense if \( N \) is really a constant (cannot change it!)

Doesn't go to \( \infty \).
Example: $u_n = (n-1)^2$: goes to $\infty$ (easy to check; do it using the def.)

$u_0 = 0$
$u_1 = 0$
$u_2 = 1$
$u_3 = 4$

$u_2 > u_1$ is true, so not quite increasing.
$u_1 > u_0$ is false. (usually don't have $u_0$!)

Limit $\lim n \to \infty$ doesn't depend on what happens for the first few members.

Example: $u_n = n + (-1)^n$ — “oscillates a bit,” gets large.

Or: $u_n = n^2$ if $1000/n$
$n^2 - 1$ if $1000/n$

In the previous example, $u_n$ was “eventually increasing.”

If $u_n \leq u_m$ for all $m < n$ then $u_m \leq u_n$ for all $m < n$.

In this example, such $m$ does not exist!

Both examples work and illustrate different points about sequences.
(e) TRUE by definition:

\[ \exists m, \forall n \geq m, \quad un \geq 1000. \]

To disprove an existential statement, we prove its negation.

Given \( x \), need \( y \) s.t. \( x^2 - y \geq 0 \)

"for all \( x \)"

Take \( y = x^2 \)

get \( x^2 - x^2 = 0 \)

#3 \( \forall y, \exists x : x^2 - y < 0 \)

FALSE.

To prove this universal statement is FALSE, we just need a counterexample:

a \( y \) such that there is no \( x \) s.t.

\[ x^2 - y \leq 0 \]

Take \( y_0 = -1 \) (any negative number would work)

Then for all \( x \in \mathbb{R} \),

\[ x^2 - y_0 = x^2 + 1 > 0 \]

so \( y_0 \) works.
Induction: (Chapter 10) - method of proof of statements about sequences, or generally statements indexed by natural numbers:

\[ P(n) \] - open sentence with \( n \in \mathbb{N} \).

\( P \) could be about anything but depends on \( n \in \mathbb{N} \).

- base case: (usually \( n = 1 \))
- prove induction step: \( P(n) \Rightarrow P(n+1) \)
- conclude by "the principle of mathematical induction" that \( P(n) \) holds for all \( n \).

**Examples:** \( P(n) \): sum of all odd numbers equals \( n^2 \)

**Pf:** base case: \( n = 1 \). odd natural numbers: \( 1 \)

\[ 1 = 1 = 1^2 \]

induction step: Need to prove: \( P(n) \Rightarrow P(n+1) \)

Assume \( P(n) \). Assume we know:

\[ 1 + 3 + 5 + \cdots + (2n-1) = n^2 \]

first \( n \) odd number

Need to prove:

\[ 1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = (n+1)^2 \]

first \( n+1 \) odd numbers.

Will finish next class. Please read it in the book!