Worksheet 8: Proof or disproof?

Decide whether the following statements are True or False, and prove/disprove them:

1. Let $\{u_n\}$ be a sequence of real numbers. Reminder: we say that $u_n \to +\infty$ if $\forall N \exists m, (n \geq m \Rightarrow u_n > N)$.

the definition →

$N \in \mathbb{N}$ - a large number.

(a) If $\cancel{\{}u_n\cancel{\}} \to \infty$, then $-2u_n \to -\infty$.

$\{u_n\}$ is a sequence s.t. $u_n \to \infty$

$\boxed{\infty \text{ is } +\infty}$

TRUE. see next page (p.2)

(b) If $u_n \to \infty$ then there does not exist m such that $u_m < 0$.

FALSE. Disproof: find a counterexample

see p. 3

(c) If $u_n \to \infty$ then $\forall n$ ~~$u_{n+1} \geq n$~~ $u_{n+1} \geq u_n$.

If $u_n \to \infty$, then it is increasing.

FALSE see p. 4

(d) The converse of (c). (First state this converse).

If $\forall n$, $u_{n+1} \geq u_n$, then $u_n \to \infty$ – the converse.

FALSE. Counterexample: $u_n = 1 - \frac{1}{n}$

increases, but stays less than 1 (~~we that~~ doesn't go to $+\infty$!)

(e) If $u_n \to \infty$ then $\exists n$ $u_n > 1000$.

TRUE see p. 5

existential statement →

2. $\exists x, \forall y$ $x^2 - y < 0$.

FALSE see p. 6

Negation: $\forall x \; \exists y : \; x^2 - y \geq 0$.

3. $\forall y, \exists x$ $x^2 - y < 0$.

FALSE see p. 6

no logical relation between (2), (3).

Pf of 1(a). Let $u_n \to +\infty$.

sequence: doesn't have to have a formula

$u_1 = 1$, ← any number

$u_2 = \pi$

$u_3 = e^3 + 2$

$u_4 = -7$

$u_5 = 1001$

$\vdots$

$n \to$ give a u_n then you have a sequence.

Then, by definition,

$$\boxed{\forall N \quad \exists m : \quad n \geq m \Rightarrow u_n > N}$$

Want to prove: first need a def'n of what it mean that $v_n \to -\infty$.

Make this definition:

$$\forall \underset{\mathbb{R}}{N} \quad \exists m \quad n \geq m \Rightarrow v_n < N$$

(to emphasize, could write:

$$\forall N > 0 \quad \exists m, \text{ s.t. } n \geq m \Rightarrow v_n < -N$$)

(The numbers v_n have to get "large negative" eventually).

Want to prove: Let $v_n = -2u_n$.

$$\forall N > 0, \exists m : \quad n \geq m \Rightarrow v_n < -N$$

Let N be given.

we want: $v_n < -N$, i.e. $-2u_n < -N$

$$\Leftrightarrow u_n > \boxed{\frac{N}{2}}$$

By definition, exists m such that

$\forall n \geq m$, we have $u_n > \frac{N}{2}$

Then for this m, we'll have $v_n < -N$, so we are done.

(b) Want an example of a sequence $\{u_n\}$ such that:

- $u_n \to \infty$ and exists m s.t. $u_m < 0$.

Examples: $-1, 0, 1, 2, 3, \ldots, n, \ldots$

$$u_n = n - 2$$

Comment: To disprove we could try to prove the negation is true:

"$u_n \to \infty$ doesn't imply that there are no negative u_m"

there exists a sequence $\{u_n\}$ s.t. $u_n \to \infty$ and $\exists\, m: u_m < 0$.

why did we put this here? ← because our statement was about all sequences even though we did not write that explicitly.

(c) "False because it just has to 'stay large', doesn't have to increase.

Use our intuition to help us find a counterexample

example: $u_n = n^3 - 5n^2$

To prove it works, we need to prove: 1) $u_n \to \infty$

2) $\exists n$ s.t. $u_{n+1} < u_n$

will require work.

(given N, you have to find m s.t.

$\forall n \geq m, \quad n^3 - 5n^2 > N$

Ok counterexample but it gives us more work than we want.

Here I asked for students' suggestions for simpler counterexamples. They are discussed below

Easier: suggestions

suggestion: $u_n = \begin{cases} n & \text{if } n \text{ is even} \\ -n & \text{if } n \text{ is odd.} \end{cases}$

does it work? $\quad -1, 2, -3, 4, -5, 6 \ldots$

unfortunately doesn't work:

we need: given N, for all $n \geq m$ that we find

need $u_n > N$

need it for all $n \geq m$ not just even n.

suggestion: $u_n = n^2 \quad u_n \to \infty$ is easy to check

unfortunately it is increasing so gives no counterexample.

suggestion: $u_n = N \leftarrow$ constant sequence only makes sense if N is really a constant (cannot change it!)

doesn't go to ∞.

Example $u_n = (n-1)^2$: goes to ∞ (easy to check do it using the def!)

$u_0 = 1$

$u_1 = 0$

$u_2 = 1$

$u_3 = 4$

...

$u_2 > u_1$ is false. so not quite increasing. could fix it, e.g. take $u_n = (n-2)^2$.

$u_1 \geq u_0$ is false. (usually don't have u_0!)

$\boxed{\text{Limit is } +\infty \text{ — whether this statement is true doesn't depend on what happens for the first few members.}}$

Example: $u_n = n + (-1)^n$ ← "oscillates a bit, gets large.

or: $u_n = n^2$ if $1000 \nmid n$

$n^2 - 1$ if $1000 \mid n$

in the previous example, u_n was "eventually increasing":

$\exists\, m : u_{n+1} \geq u_n$ for all $n \geq m$

In this example, such m does not exist!

Both examples work and illustrate different points about sequences.

(e) TRUE by definition:

~~$\exists$ for all u_n with~~

$$\exists m, \quad \forall n \geq m, \quad u_n > 1000.$$

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2 To ~~prove~~ disprove an existential statement, we prove its negation.

Given x, need y s.t. $x^2 - y \geq 0$

"for all x"

Take $y = x^2$

get $x^2 - x^2 = 0$

#3 $\forall y, \exists x : x^2 - y < 0$

FALSE.

To prove this universal statement is FALSE, we just need a counterexample:

a y such that there is NO x s.t.

$$x^2 - y < 0$$

Take $y_0 = -1$ (any negative number would work)

Then for all $x \in \mathbb{R}$,

$$x^2 - y_0 = x^2 + 1 > 0,$$ so y_0 works.

Induction: (Chapter 10).

- method of proof of statements about sequences, or generally statements indexed by natural numbers:

$P(n)$ - open sentence with $n \in \mathbb{N}$.

could be about anything but depends on $n \in \mathbb{N}$.

- base case: (usually $n=1$)
- prove induction step: $P(n) \Rightarrow P(n+1)$
- conclude by "the principle of mathematical induction" that $P(n)$ holds for all n.

Examples: $P(n)$: sum of ~~all~~ the first n odd natural numbers equals n^2

$\underbrace{1+2+\dots+(2n-1)}_{\text{first } n \text{ odd numbers}}$

Pf: base case: $n = 1$. odd natural numbers: $\underset{\text{when } n=1}{2n-1} = 1$, that's all.

$$\underset{\text{the sum}}{1} = \underset{n^2}{1}$$

induction step: Need to prove: $P(n) \Rightarrow P(n+1)$

Assume $P(n)$. Assume we know:

induction assumption:

$$1+3+5+\dots+\underset{\text{the } n\text{th odd number}}{(2n-1)} = n^2$$

Need to prove: $\underbrace{1+3+5+\dots+(2n-1)+(2n+1)}_{\text{first } n+1 \text{ odd numbers.}} = (n+1)^2$

Will finish next class. Please read it in the book!