Notes on indexed collections of sets, and quantifiers.

The main point of this note is to emphasize the relationship between open sentences and sets: let $P(x)$ be an open sentence, where the domain for the variable x is a set U. We think of U as the universal set. Then, we can associate a subset of U with the open sentence P – it is the set A_P defined as $A_P := \{x \in U \mid P(x) \text{ is true}\}$. Thus, every open sentence *partitions* the domain into two disjoint sets: the set A_P of all elements x such that $P(x)$ is true, and the set of all x such that $P(x)$ is false. Conversely, if we start with sets, then every subset A of the universal set U corresponds to the open sentence '$x \in A$'.

With this correspondence between sets and sentences, conjunction of statements corresponds to the intersection of sets, disjunction corresponds to union of sets, and negation corresponds to taking the complement. (Think about this and make sure it makes sense).

Now we can give a precise definition for the union and intersection of an indexed collection of sets.

Suppose we have a collection of sets A_s, indexed by elements $s \in S$ of some set S. For example, if $S = \{1, 2\}$, then we only have two sets A_1 and A_2. If $S = \mathbb{N}$ – the set of natural numbers, then $\{A_n\}_{n\in\mathbb{N}}$ is an infinite collection of sets $A_1, A_2, \dots A_n, \dots$. If $S = \mathbb{R}$ is the set of all real numbers, it means we have a set A_r for every number $r \in \mathbb{R}$. For example, we can consider the collection $A_r = [0, r] \times [0, r] \subset \mathbb{R} \times \mathbb{R}$ of squares of size r on the plane – for every positive real number r, we have a square A_r. In this example, the indexing set S is the set of all positive real numbers: $S = \{r \in \mathbb{R} \mid r > 0\}$.

How to define the union of an indexed collection of sets? The union has to be the set of elements contained *in at least one of the sets* of the collection.

Definition 1. Let $\{A_s\}_{s\in S}$ be an indexed collection of sets, indexed by the elements of some set S. Then the union of this collection is the set

$$\bigcup_{s\in S} A_s = \{x \mid \exists s \in S, \text{ such that } x \in A_s\}.$$

On the other hand, the intersection of sets is the set of their *common elements* – so it has to be the set of elements that belong to *all* the members of our collection of sets. So we arrive at the definition of the intersection:

Definition 2. The intersection of the collection A_s is the set

$$\bigcap_{s\in S} A_s = \{x \mid \forall s \in S, x \in A_s\}.$$

Note that these definitions (of course) agree well with DeMorgan laws: the complement of the union should be the intersection of complements. Let us see why this holds for indexed collections as well:

$$\overline{\bigcup_{s\in S} A_s} = \overline{\{x \mid \exists s \in S, \text{ such that } x \in A_s\}}$$

$$= \{x \mid \nexists s \in S, \text{ such that } x \in A_s\} = \{x \mid \forall s \in S, x \notin A_s\} = \bigcap_{s\in S} \overline{A_s}.$$

Exercise: make sure you understand every equality above.