Review session: Mon Apr 13, 2-4 pm. (?) (poll on Piazza) maybe.

Remember: Teaching Evalu.

Today: Cardinality. (Review).

Infinite sets:
- Continuum: \( \mathbb{R} \)
- "unnaturally large": \( \mathcal{P}(\mathbb{R}) \)
- Countably infinite: \( \mathbb{N}, \mathbb{Q} \)

Worksheet 20: 1. Want to prove: \( A_1, \ldots, A_n \) countable

Then \( A_1 \times \cdots \times A_n \) is countable.

- Let \( \ell \) be finite, then the product is finite,
  \( |A_1 \times \cdots \times A_\ell| = |A_1| \cdot |A_\ell| = |A_\ell|. \)
- If at least one \( \beta \) is infinite, then the product
  \( \beta \) infinite, we need to prove it is countable.

Proof by Induction: (all are countably infinite, then product is countably infinite)

Base: \( n = 1, |A_1| \), nothing to prove.

Induction step: Assumption: \( A_1, \ldots, A_k \) count.

\( \Rightarrow A_1 \times \cdots \times A_k \) is count. \( \forall \).

Need to prove: \( A_{k+1} \) also countably \( \forall \).

\( \Rightarrow A_1 \times \cdots \times A_k \times A_{k+1} \)

\( \exists \) also count. \( \forall \).
Lemma: \( A, B \) - count. \( \Rightarrow \) \( A \times B \) is countably infinite

(we proved: \( \mathbb{N} \times \mathbb{N} \) is countably infinite)

\[ f: \mathbb{N} \rightarrow A \quad \text{bijective} \]
\[ g: \mathbb{N} \rightarrow B \quad \text{bijective} \]

Then \( h: \mathbb{N} \times \mathbb{N} \rightarrow A \times B \quad \text{bijective} \)
\[ h(m, n) = (f(m), g(n)) \]

Then we take \( A = A_1 \times \ldots \times A_n \) count. by the induct. assum.

By Lemma, \( A \times B \) is countably infinite, which completes pf of induction step.

Last thing: What if some of our sets are finite, and some are countably infinite?

Lemma: "we can permute the factors":

\[ |A \times B| = |B \times A| \]

Pf: Need \( f: A \times B \rightarrow B \times A \) bijective.

Define \( f(a, b) = (b, a) \)

easy exer: it is bijective.

This lemma allows us to collect the factors:

\[ A_1 \times \ldots \times A_n = (A_1 \times \ldots \times A_n) \times (A_j \times \ldots \times A_j) \]

\[ \text{(finite ones)} \quad \text{(all infinite)} \]

\[ \text{(finite, call it } B) \quad \text{(countably infinite, call it } C) \]

Last lemma: if \( B \) is finite, \( C \) countably infinite, then
B \times C is countably infinite.

**Pf:** Suppose \( n = 1 \mathbb{B} \). Let \( B = \{ b_1, \ldots, b_n \} \).

Then \( B \times C = C_1 \cup C_2 \cup \ldots \cup C_n \).

Let \( C_i = \{ (b_j, c) \mid c \in C \} \).

Think of it as a copy of \( C \).

\[ C = \mathbb{R} \]

Each \( C_i \) is a copy of the line \( C \).

Lemma: \( A_1, A_2 \) - countably infinite \( \Rightarrow A_1 \cup A_2 \) is countably infinite.

\[ f_1 : \mathbb{N} \rightarrow A_1 \]
\[ f_2 : \mathbb{N} \rightarrow A_2 \]

(same as \( f \) that \( |\mathbb{Z}| = |\mathbb{N}| \).)

Then by induction, \( A_1 \cup A_2 \cup \ldots \cup A_n \) is countably infinite if \( A_1, \ldots, A_n \) is countably infinite.

We proved \( |\mathbb{Z}| = |\mathbb{N}| \).

This finishes the proof that \( B \times C \) is countably infinite.

Define \( f : \mathbb{Z} \rightarrow A_1 \cup A_2 \)

\[ f(n) = \begin{cases} f_1(n) & \text{if } n > 0 \\ f_2(-n) & \text{if } n \leq 0 \end{cases} \]

This function is surjective but may not be injective if \( A_1 \cap A_2 \neq \emptyset \).

Then by Problem 3, \( A_1 \cup A_2 \) will be countably infinite.
Problem 2, 3: \( f: A \to \mathbb{N} \) - injective. (both are Theorems in the text)

Then \( A \) is finite or countably infinite.

Pf: \( 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \ldots \)

Elements of \( f(A) \).

Since \( f \) is injective, \( f: A \to f(A) \subset \mathbb{N} \) is bijective.

So we just need to prove that \( f(A) \) is countable.

So we need to give a method for "numbering" the elements of \( f(A) \).

Take the smallest one, call it \( b_1 \).

The next one is \( b_2 \).

This ends if \( f(A) \) is finite.

Otherwise we have a way of labelling elements of \( f(A) \) by natural numbers, so it is countably infinite.

Remark: to make this more precise:

axioms & natural numbers:
include "axiom of induction":

\( \text{"obvious" about } \mathbb{N}. \text{ axiom!} \)

\( \text{every non-empty subset of } \mathbb{N} \) has the smallest element

\( \text{\( \mathbb{N} \) is well-ordered set.} \)

\( \text{\( \mathbb{N} \) is well-ordered set.} \)

\( \text{(notes } \mathbb{Z}, \text{ or } \{ a \in \mathbb{Q} : a > 0 \} \text{ do not satisfy this... It is a very strong statement!)} \)
Our proof can be made more precise if we refer to this axiom:

Let \( b_1 = \text{smallest elt of } f(A) \) (exists by the axiom)

\( b_2 = \text{smallest elt of } f(A) \setminus \{b_1\} \)

\[ \vdots \]

\( b_n = \text{smallest elt of } f(A) \setminus \{b_1, \ldots, b_{n-1}\} \)

for every \( n \).

So we established a bijection between \( f(A) \) and \( \mathbb{N} \), and \( f \) gives a bijection between \( A \) and \( f(A) \), so we get

\[ |A| = |f(A)| = |\mathbb{N}|. \]

Problem 3: \( f : \mathbb{N} \rightarrow A \) surjective, then \( A \) is countable.

Try this: let \( a_1 = f(1) \). Let \( a_2 = f(2) \) s.t. \( f(2) \neq f(1) \)

If \( f(2) = f(1) \),

let \( a_2 = f(k) \), \( k = \text{smallest natural number} \)

s.t.

\( f(k) \neq f(1) \)

What if \( \{ k \in \mathbb{N} : f(k) \neq f(1) \} \)

If this set is empty, is \( f(k) = f(1) \) for all \( k \) in \( \mathbb{N} \)?

Then \( A = \{ a_1 \} = \{ f(1) \} \) b/c \( f \) is surjective.

Then \( A \) is a set of one element.
So: we have $a_1$, and we defined $a_2$ or
proved that $|A| = 1$.

Proceed as before:

$a_3 = f(k_3)$, where $k_3$ is smallest natural
number s.t.

$f(k_3) \neq a_1$ or $a_2$.

again, if such $k_3$ doesn't exist,

then $|A| = 2$ (then $A = \{a_1, a_2\}$).

$\text{b/c } f$ is surjective.

If this process ends, $A$ is finite.

if not, $A$ is countably infinite.

(every element of $A$ is labelled, b/c

$f$ is surjective).

Remark: in both problems, we “do not need
all of $N$” to label elements of $A$.

Problem 4: $A = \{a_1, a_2, \ldots, a_n, \ldots\}$?

\[ A_n = \{a_1^{(n)}, a_2^{(n)} \ldots\} ? \]

helpful: group the finite ones together into
one finite set, and prove:

1) if each $A_1, \ldots, A_n$ is countably infinite
then the union is countably infinite.

2) $A$ - countably infinite, $B$ is finite, $A \cap B$
then $B \cup A$ is countably infinite.
Pf. of (1): arrange them in a table, as before:

\[ a_i, a_i, \ldots \quad \text{-- set } A_1 \]
\[ a_i, a_i, \ldots \quad \text{-- set } A_2 \]

Make a "snake path" visiting them all, as in the proof of \( |N \times N| = |N| \).

(or just say: let \( f : N \times N \to \bigcup_{i=1}^{\infty} A_i \) be the function defined by \( f(m, n) = a_{i(m)} \).)

Then \( f \) is surjective (note: might not be injective)

so we have \( f : N \times N \to \bigcup_{i=1}^{\infty} A_i \); surjective.

Since \( |N \times N| = |N| \), by the previous problem, \( |\bigcup_{i=1}^{\infty} A_i| = |N| \).

Problem 5 practice!

6 - easy \(<\) do it!

Problems 7, 8: \( \& \) follows from 7.

\#7: Trick: make any countably infinite subset,

\[ e.g. \quad A = \{ \frac{1}{n} \mid n \in N \} \subset (0,1) \]
\( (0, 1) = A \cup B, \text{ where } B = (0, 1) \setminus A. \)

\( \{0, 1\} = \{0\} \cup \{1\} \cup \{0, 1\}. \)

Now define \( f: (0, 1) \rightarrow (0, 1) \) piecewise:

\[
f(x) = \begin{cases} 
  x & \text{if } x \in B \\
  g(x) & \text{if } x \in A
\end{cases}
\]

where \( g \) is a bijective function between \( A \) and \( A \cup \{0, 1\} \).

\[\text{ex: } A = \{\frac{1}{n} | n \in \mathbb{N}\}\]

Define:

\[
g\left(\frac{1}{n}\right) = \begin{cases} 
  \frac{1}{n-1} & \text{if } n \neq 1 \\
  0 & \text{if } n = 1
\end{cases}
\]

You can always add / throw away a finite set from an infinite set, and the cardinality doesn’t change.

Also, can add / subtract a countable set from an uncountable set, and cardinality doesn’t change.

9–11: last problem of HW 12.

Sequences of 0s and 1s encode IR ("binary":)

\[
\begin{array}{cccccccccccc}
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]