Math 220 Workshop 5: Cardinality.

1. Let $A_1, \ldots, A_n$ be denumerable sets. Prove that $A_1 \times \cdots \times A_n$ is denumerable.
   \textbf{Hint.} Use induction.

2. Prove that if $A_n$ is countable for all $n \in \mathbb{N}$, then $A = \bigcup_{n=1}^{\infty} A_n$ is also countable. You may assume each $A_n$ is non-empty (or just leave it out).
   \textbf{Hint.} Try to arrange the elements of $A$ in a table, then use it to define a function $f : \mathbb{N} \times \mathbb{N} \to A$, and show this function is onto. Why does the result follow?

3. Let $B$ be a denumerable set, and let $f : A \to B$ be a surjective function such that $f^{-1}(x)$ is a countable set for every $x \in B$. Prove that $A$ is denumerable.
   \textbf{Hint.} Use the previous problem.

4. Prove that $|(0, 1)| = |[0, 1]|$.
   \textbf{Hint.} Choose a denumerable subset $A$ of $(0, 1)$. Then make a bijection between $A$ and $A \cup \{0\}$. Then define your function on the rest of the interval.

5. (a) Prove that if $A$ is a denumerable subset of real numbers, then $|\mathbb{R} - A| = |\mathbb{R}|$.
   \textbf{Hint.} Modify the previous hint as follows. Choose $b_n \in (n, n + 1) - A$ for all $n \in \mathbb{N}$. (Why does such a $b_n$ exist?) Hence $B = \{b_n : n \in \mathbb{N}\}$ is a denumerable set of reals disjoint from $A$. Now choose a bijection $f : A \cup B \to B$. (Explain why this exists using an earlier question.) Finally extend $f$ to a bijection from $\mathbb{R}$ to $\mathbb{R} - A$.

(b) Prove that $|\mathbb{I}| = |\mathbb{R}|$, where $\mathbb{I}$ is the set of irrationals as usual.