Questions discussed in class on September 23.

The task at the end of class was: write the following statements using logic symbols, and negate them.

(1) All my friends are older than me.
(2) Whenever the input sequence is recursive, the Turing machine halts.
(3) Every polynomial with complex coefficients has a root.
(4) $f$ is an isomorphism if it is surjective or injective.
(5) Baby monsters are blue until they grow tails.

Solutions:

(1) Let $F$ be the set of my friends. Let $P(x)$ be the open sentence (where $x \in F$), stating "$x$ is older than me". Then the symbolic expression for this statement is:

$$\forall x \in F, P(x).$$

The negation:

$$\exists x \in F, \sim P(x).$$

In words, it says: "Not all my friends are older than me", or: "there exists a friend who is not older than me", or "I have a friend who is not older than me (younger or same age)."

(2) Let $R$ be the statement "The input sequence is recursive", and let $H$ be the statement "Turing machine halts". Then we can write our statement as $R \Rightarrow H$. Notice that here it is implied that this holds for every recursive sequence (whatever this whole sentence means). Please read Sections 2.8 and 2.9 about such implied quantifiers! Alternatively, we could write: let $S$ be the set of all possible input sequences, let $R(s)$ be the open sentence "$s$ is recursive", and let $H(s)$ be the open sentence "Turing machine halts with the input sequence $s". Then our statement says:

$$\forall s \in S, R(s) \Rightarrow H(s),$$

which can be written as a shorthand:

$$R(s) \Rightarrow H(s).$$

The negation (the long form is the easiest to negate):

$$\exists s \in S : R(s) \land \sim H(s),$$

which in words says: "There exists an input sequence $s$ which is recursive and such that the Turing machine does not halt."

(Recall the rule for negating the implication!)
(3) Let $P$ be the set of all polynomials with complex coefficients, let $R(p)$ be the open sentence "$p$ has a root". Then our statement is: $\forall p \in P, R(p)$. The negation: "Not every polynomial with complex coefficients has a root". This statement has very similar structure to Example (1) above – compare them!

(4) Let $P$ be the statement "$f$ is an isomorphism", let $Q$ be the statement "$f$ is surjective", and let $R$ be the statement "$f$ is injective". Then our statement says

$$(Q \lor R) \Rightarrow P.$$  

The negation is:

$$(Q \lor R) \land \sim P,$$

which in words says, "It is possible that $f$ is surjective or injective, but it is not an isomorphism".

(5) Our statement can be rephrased as "If a baby monster doesn’t yet have a tail, then it is blue", so if $P(m)$ is the open sentence "baby monster $m$ has a tail", and $Q(m)$ is the open sentence "$m$ is blue", then our statement is:

$$\sim P(m) \Rightarrow Q(m).$$  

(again, note that "for all baby monsters $m$" is implied in this sentence). Its negation is "there exists a baby monster that has no tail and is not blue".