Notes on Review Session Dec 16^{th}, 2016

* Main Topics :
  * congruences
  * induction
  * functions
  * proofs about sets
  * cardinality
  * equivalence relations
  * logic

Induction :

Question 3 : 2014 (Dec)

$x \in \mathbb{R}$, $x \geq 3$ use induction to prove :

For all $n \in \mathbb{N}$, $\frac{(n+2)(n+1)n}{6} x^3 \leq (1+x)^{n+2}$

Do not do induction on the "x"!

Induction ~~of~~ on ~~x~~ n !

(x is a constant parameter.) ← Haven't prove yet.

Base Case : $n=1$ $\quad \frac{3 \cdot 2 \cdot 1}{6} \cdot x^3 \overset{?}{\leq} (1+x)^{1+2}$

$\iff x^3 \leq (1+x)^3$ true

Keep biconditional

Induction step : not "all" n = k (for some k) .

Assume true for n = k prove for n = k+1.

How to get from here to

Assumption :

$\frac{(k+2)(k+1)k}{6} x^3 \leq (1+x)^{k+2}$

→ To prove : $\frac{(k+3)(k+2)(k+1)}{6} x^3 \leq (1+x)^{(k+3)}$

Scratch work : Trying to Divide

Want : $\frac{(k+3)(k+2)(k+1)}{(k+2)(k+1)k}$ ← would like $\leq 1+x$

have

if this were true combine with induction

Scratch work cont.:

Very honestly check if it were true

would like : $\frac{(k+3)}{k} \overset{?}{\leq} 1+x$

make sure biconditionals are true

$\cancel{1}+\frac{3}{k} \overset{?}{\leq} \cancel{1}+x$

$\frac{3}{k} \leq x$

watch out dividing 0's!)

know that : $x \geq 3$ TRUE!

$k \geq 1$

our statement is equivalent to the last statement.

If you are desperate (1 min left etc.) just say everything is biconditional. Hence true

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actual Proof of ind. step:

Since $k \geq 1$ and $x \geq 3$, we have $\frac{3}{k} \leq x$,

Then $1+\frac{3}{k} \leq 1+x$

Then $\frac{k+3}{k} \leq 1+x$

and we get $\frac{(k+3)(k+2)(k+1)}{(k+2)(k+1)k} \leq 1+k$

multiply by induction assumption and get what we need to prove.

(note: both sides are positive).

■

B Everything on Summation and Fibonacci are fair game.

(note: ~~minimal~~ smallest counterexample.

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slight change → Dec 2011 #6

$\sum_{k=1}^{n} (2k-1)$ is a square. Prove by induction.

Induction Proof:

Scratch work to gain confidence

1) Try to guess RHS. [Try out the first few n's might help

n=1 Plug in k=1

sum has only only term.

$2\cdot 1-1=1=1^2$

LHS = 1 = R.

n=2 sum has 2 terms

$k=1$, $k=2$

$k=1$ $2.1-1=1$

$k=2$ $2.2-1=3$

} our sum is $1+3=4=2^2$.

n=3 Sum has 3 term

$k=1$ $2\cdot 1-1=1$

$k=2$ $2.2-1=3$

$k=3$ $2\cdot 3-1=5$

$1+3+5=9=3^2$

Oh! we realize ~~we~~ $1^2, 2^2, 3^2 \ldots$ well prove:

$$\sum_{k=1}^{n}(2k-1)=n^2$$

<u>Induction step</u>:

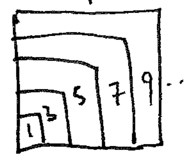
Assume true for $n=m$, prove: for $n=m+1$

<u>given</u>: $\sum_{n=1}^{m}(2k-1)=m^2$

<u>To prove</u>: $\sum_{k=1}^{m+1}(2k-1)=(m+1)^2$

$$\underbrace{\overset{k=1}{1}+\overset{k=2}{3}+\overset{k=3}{5}+\cdots+\overset{k=m}{(2m-1)}}+\overset{k=m+1}{(2(m+1)-1)} \overset{?}{=} (m+1)^2$$

n^2



recall: in class drawing

← people should know

By induction assumption:

$$LHS = m^2 + 2(m+1) - 1$$
$$= m^2 + 2m + 2 - 1$$
$$= m^2 + 2m = (m+1)^2$$

[if the equality is True, algebra works out. [Have faith!]
if algebra does not works, go and check if the base case!
and check induction step.
Do not say everything works when it doesn't!

Induction in cardinality proof:

Example: 1) Given $A_1 \dots A_n$, and $B_1 \dots B_n$ are sets and
$|A_i| = |B_i|$ for all i

induction on n Prove: $|A_1 \times \dots \times A_n| = |B_1 \times \dots \times B_n|$)

Base Case: $n=1$ $|A_1| = |B_1|$ - given tautology!

btw (It's okay to check more than 1 base case!)

$n=2$: $|A_1| = |B_1|$ and $|A_2| = |B_2|$.

Prove $|A_1 \times A_2| = |B_1 \times B_2|$ (never say anything about same element! Bijection!)

Given: $f_1: A_1 \to B_1$

aside: $f(x)$ - the name of the element of B.
$x \in A$. Don't find the relationship or formula.

Given: $f_1: A_1 \to B_1$ bijective!
$f_2: A_2 \to B_2$

Need to construct a bijective function ~~from~~

$g_1 : A_1 \times A_2 \longrightarrow B_1 \times B_2$.

Define: $g(x_1 x) \quad \cancel{} = (f_1(x_1), f_2(x_2))$

where $(x_1\ x_2) \in A_1 \times A_2$ $\quad \uparrow B_1 \times B_2$.

so $x_1 \in A_1$, $y_2 \in A_2$.

Let ~~suppose~~ us prove that g is bijective :

1) Injective $\quad$ or $\quad$ inverse exists $\leftarrow$ too hard and easy mistakes

2) Surjective

→ ~~preff~~ preferred $\quad$ ① Suppose $(x_1, x_2) \in A_1 \times A_2$ $\quad x_1 \neq y_1$ or

and $(y_1, y_2) \in A_1 \times A_2$ and $x_2 \neq y_2$

$g(x_1, x_2) = (f_1(x_1), f_2(x_2))$.

$g(y_1, y_2) = (f_1(y_1), f_2(y_2))$.

we know : $x_1 \neq y_1$ or $x_2 \neq y_2$.

WLOG say $x \neq y_1$ then $f_1(x_1) \neq f_1(y_1)$.

Then $g(x_1, x_2)$ has different first coordinate from $g(y_1 y_2)$.

They are different

② Surjective :

let $(b_1, b_2) \in B_1 \times B_2$

Need to prove : $\exists\ (x_1, x_2) \in A_1 \times A_2$ s.t. $g(x_1\ x_2) = (b_1, b_2)$

By definition $g(x_1, x_2) = (f_1(x_1), f_2(x_2))$

So we want $(f_1(x_1), f_2(x_2)) = (b_1, b_2)$

since f_1 ~~and~~ is surjective,

exists $x_1 \in A_1$ st.

$f_1(x_1) = b_1$

Similarly $\exists x_2 \in A_2$ st $f_2(x_2) = b_2$

Then $g(x_1, x_2) = (b_1, b_2)$ ■

Induction Step

given $|A_i| = |B_i|$ for $i = 1, 2, \dots, n+1$

want to prove:

$$|A_1 \times \dots \times A_{n+1}| = |B_1 \times \dots \times B_{n+1}|$$

can assume: $|A_1 \times \dots \times A_n| = |B \times \dots \times B_n|$.

Okay not really but a much better way

$$\begin{array}{ccc} (a_1, \dots, a_{n+1}) & & ((a_1, \cdots a_n), a_{n+1}) \\ A_1 \times \dots \times A_n \times A_{n+1} & = & (A_1 \times \dots \times A_n) \times A_{n+1} \end{array}$$

we can define a bijective function from $A_1 \times \dots \times A_{n+1}$ TO $(A_1 \times \dots \times A_n) \times A_{n+1}$

$$f(a_1 \dots a_{n+1})) = ((a_1 \dots a_n), a_{n+1})$$

$$|A_1 \times \dots \times A_{n+1}| = |(A_1 \times \dots \times A_n) \times A_{n+1}|$$

Same for B's.

let $C = A_1 \times \dots \times A_n$

$D = B_1 \times \dots \times B_n$

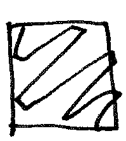
By induction assumption: $|C| = |D|$

$$|C \times A_{n+1}| = |D \times B_{n+1}|$$

This is $n = 2$ case. ■

Fact: $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

Proof: arrange in a table walk around it.




Problem:

$a_1, \dots a_n$ countable, prove:

$A_1 \times \cdots \times A_n$ is countable infinite

Induction: $n=1$ –

$n=2$ $|A_1 \times A_2| = |\mathbb{N} \times \mathbb{N}|$

(needs proof which we just did in pg 5).

$|A_1 \times \cancel{B_2}\, A_2| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$, prove in class

Induction Step:

$|A_1 \times \cdots \times A_{n+1}|$

$= |(A_1 \times \cdots \times A_n \times A_{n+1}|$

$= |\mathbb{N} \times A_{n+1}|$

(by induction assumption and the $n=2$ case)

$= |\mathbb{N} \times A_{n+1}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

April 2013:

$|A| = |B|$ Then $|P(A)| = |P(B)|$

only prove it for infinite [wrong way: $|A| = |B|$,

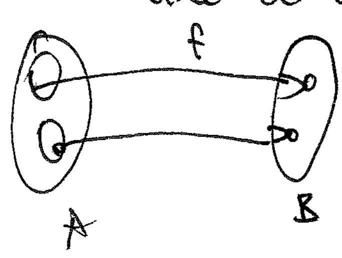
$|P(A)| = 2^{|A|}$

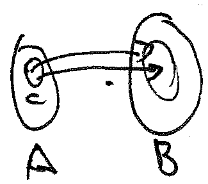
$2^{|A|} = 2^{|B|}$ – just restating the problem.]

Given: $\exists f: A \to B$ [Don't do contradiction]

Need to prove exists $g: P(A) \to P(B)$

Think to make a bijective $g: P(A) \to P(B)$





Input: a subset $C \subseteq A$

let $C \subseteq A$, we need to define $g(C)$, which should be a subset of B.

Define $g(C) = \{f(x) : x \in C\} \subseteq B$

few examples:

Take $C = A$,

$g(C) = B$ (because f is surjective)

Prove: g is bijective

injective: Take $C_1, C_2 \subseteq A$, $C_1 \neq C_2$.

Need to prove: $g(C_1) \neq g(C_2)$.

$C_1 \neq C_2$ means: $\exists x_0 \in C_1$ and $x_0 \notin C_2$ or

$\exists x_0 \in C_2$ and $x_0 \notin C_1$

WLOG say $x_0 \in C_1$ $x_0 \notin C_2$

Need to prove: $g(C_1) \neq g(C_2)$

This means

$\{f(x) : x \in C_1\} \neq \{f(x) : x \in C_2\}$.

We prove: $f(x_0) \in f(C_1)$, $f(x_0) \notin f(C_2)$.

We have: $x_0 \in C_1$ so $f(x_0) \in f(C_1)$,

(By def of image of a set.)

want to prove: $f(x_0) \notin f(C_2)$. (This means there is no element y of C_2, s.t $f(x_0) = f(y)$)

This is true because f is injective

$g(C_1) = f(C_1) = \{f(x) : x \in C_1\}$

image of subset.

C_1 is an element of $P(A)$

Everyone is happy with surjectivity

Note: we just proved if you take any finite number of countable sets $|A_1 \times \dots \times A_n| \Rightarrow$ countable.

But $\underbrace{\mathbb{N} \times \dots \times \mathbb{N} \cdots}_{\text{infinitely}} = \{(a_1, a_2 \dots a_n \cdots) \mid a_i \in \mathbb{N}\}$

uncountable! (cantor's argument!)

Prove:

$12 \mid n^4 - n^2$

Fail proof: consider 12 cases

$n \equiv 0 \mod 12$ $\quad 0^4 - 0$

~~$n \equiv$~~ $n \equiv 1 \mod 12$ $\quad 1^4 - 1^2$

$n \equiv 2 \mod 12$ $\quad 2^4 - 2^2$

$\vdots$ $\quad \vdots$

$11^4 \equiv (121)^2 \equiv 1^2$

11 $\quad 11^4 - 11^2 = 1^2 - 1^2 \equiv 0 \mod 12$

Solution 2: Fact: $a \mid c$

$b \mid c$

$\gcd(a,b) = 1 \Rightarrow ab \mid c$

enough to prove: $3 \mid n^4 - n^2$ and $4 \mid n^4 - n^2$

(10)

$n^4 - n^2 = n^2(n-1)(n+1)$ $\quad n \equiv 0, 1, 2, 3 \mod 4$

$[n]^4 - [n^2] = [0]$ in $\mathbb{Z}_{12}$!

notation