1. Does there exist a continuous bijective function \( f : \mathbb{R} \rightarrow \mathbb{R} - \{1\} \)? Explain.

*Hint*: Recall the Intermediate Value Theorem.

**Solution**: The answer is “No”. Suppose \( f : \mathbb{R} \rightarrow \mathbb{R} - \{1\} \) is bijective. Then there exists \( a \in \mathbb{R} \), such that \( f(a) = 0 \), and there exists \( b \in \mathbb{R} \), such that \( f(b) = 2 \). Then, since \( f \) is continuous, by the Intermediate Value Theorem, there exists \( c \in (a, b) \), such that \( f(c) = 1 \). Thus, the range of such a continuous function has to contain 1.

2. Let \( A_1, A_2, B_1, B_2 \) be non-empty sets such that \( |A_i| = |B_i| \) for \( i = 1, 2 \). Prove that

(a) \( |A_1 \times A_2| = |B_1 \times B_2| \).

(b) If \( A_1 \cap A_2 = B_1 \cap B_2 = \emptyset \), then \( |A_1 \cup A_2| = |B_1 \cup B_2| \).

Remember — the sets may or may not be finite. This also applies to the remaining questions below.

**Solution**: (a)

- Since \( |A_i| = |B_i| \) there exist bijections \( f_i : A_i \rightarrow B_i \), \( i = 1, 2 \).
- Define \( h : A_1 \times A_2 \rightarrow B_1 \times B_2 \) by \( h(a_1, a_2) = (f_1(a_1), f_2(a_2)) \). We must show it is a bijection.
- Injection. Let \( (a_1, a_2), (a'_1, a'_2) \in A_1 \times A_2 \). Assume that \( h(a_1, a_2) = h(a'_1, a'_2) \). By our definition of \( h \) we know that \( f_1(a_1) = f_1(a'_1) \) and so \( a_1 = a'_1 \). Similarly, \( f_2(a_2) = f_2(a'_2) \) and so \( a_2 = a'_2 \). Thus \( (a_1, a_2) = (a'_1, a'_2) \) and so \( h \) is injective.
- Surjection. Let \( (b_1, b_2) \in B_1 \times B_2 \). Since \( f_1 \) and \( f_2 \) are surjective there are \( a_1 \in A_1 \) and \( a_2 \in A_2 \) so that \( f_i(a_i) = b_i \), for \( i = 1, 2 \). Now \( (a_1, a_2) \in A_1 \times A_2 \) and \( h(a_1, a_2) = (b_1, b_2) \). Thus \( h \) is surjective.
- Since \( h \) is injective and surjective, it is bijective and the two sets have the same cardinality.

(b) Since \( |A_1| = |B_1| \) and \( |A_2| = |B_2| \), there exist bijective functions \( f_1 : A_1 \rightarrow B_1 \) and \( f_2 : A_2 \rightarrow B_2 \). We need to construct a bijective function \( h : A_1 \cup A_2 \rightarrow B_1 \cup B_2 \). Let us define it piece-wise:

\[
    h(x) = \begin{cases} 
        f_1(x) & \text{if } x \in A_1 \\
        f_2(x) & \text{if } x \in A_2.
    \end{cases}
\]

The function \( h \) is well-defined since the sets \( A_1 \) and \( A_2 \) do not have any common elements. It is easy to see that \( h \) is bijective. First, show that \( h \) is injective. Suppose \( h(a) = h(b) \) for some \( a, b \in A_1 \cup A_2 \). There are three cases:

- Case 1: both \( a \) and \( b \) are in \( A_1 \). Then \( h(a) = f_1(a) \) and \( h(b) = f_1(b) \) by definition of \( h \). Then, since \( f_1 \) is injective, we get \( a = b \).
• Case 2: both \( a \) and \( b \) are in \( A_2 \). This case is similar: since \( f_2 \) is injective, we get \( a = b \).

• Case 3: one of the elements is in \( A_1 \), and the other – in \( A_2 \). We denote the one that is in \( A_1 \) by \( a \). So, we have \( a \in A_1 \) and \( b \in A_2 \). Then \( h(a) = f_1(a) \in B_1 \), and \( h(b) = f_2(b) \in B_2 \), but since \( B_1 \cap B_2 = \emptyset \), in this case the equality \( h(a) = h(b) \) is impossible.

Now let us prove that \( h \) is surjective. Let \( b \in B_1 \cup B_2 \). Then either \( b \in B_1 \) or \( b \in B_2 \). If \( b \in B_1 \), since \( f_1 \) is surjective, there exists \( a \in A_1 \) such that \( f_1(a) = b \). Since \( a \in A_1 \), by definition of \( h \), we have \( h(a) = f_1(a) = b \). So, we proved that there exists \( a \in A_1 \cup A_2 \) such that \( h(a) = b \). The case \( b \in B_2 \) is similar (replace \( A_1 \) with \( A_2 \) everywhere).

3. Let \( A \) be an non-empty set. Prove that \(|A| \leq |A \times A|\).

**Solution:**

- It suffices to find an injection from \( A \) to \( A \times A \).
- Define \( f : A \to A \times A \) by \( f(a) = (a, a) \).
- Let \( a, b \in A \) and assume \( f(a) = f(b) \). Thus \((a, a) = (b, b)\) and so we must have \( a = b \). Hence \( f \) is injective.

4. Prove that if \( A \) is a denumerable set, and there exists a surjective function from \( A \) to \( B \) (and \( B \) is infinite), then \( B \) is denumerable.

**Solution:** Since \( A \) is denumerable, by definition there exists a bijective function \( f : \mathbb{N} \to A \). It is given that there exists a surjective function \( g : A \to B \). Then \( g \circ f \) is surjective function from \( \mathbb{N} \) to \( B \). (recall that a composition of two surjective functions is surjective – you should know the proof of this fact). So, it is enough to prove the following statement:

* if \( B \) is an infinite set such that there is a surjective function from \( \mathbb{N} \) to \( B \), then \( B \) is denumerable.

Proof of the statement: let \( h : \mathbb{N} \to B \) be a surjective function. We will define a subset \( A \) of \( \mathbb{N} \) such that the restriction of the function \( h \) to the set \( A \) is bijective. Then we will have that \( B \) is in bijection with an infinite subset of \( \mathbb{N} \) (namely, the set \( A \)), which is denumerable by a theorem we proved in class. So it remains to construct the set \( A \). We construct it *inductively* – this means, we will define a procedure that allows us to decide whether a number \( n \) should be in \( A \) or not, assuming we have already decided that about the numbers \( 1, \ldots, n - 1 \). We start at 1, and decide that \( A \) should contain 1. Now, assuming we already made a decision about the numbers
5. Prove that if $A$ and $B$ are denumerable sets, and $C$ is a finite set, then $A \cup B \cup C$ is denumerable.

**Note:** We use this fact, as well as the previous problem, when proving that the set of rational numbers $\mathbb{Q}$ is denumerable.

**Solution:** We will prove that there is a surjective function from $\mathbb{N}$ to $A \cup B \cup C$. Then, since this set is infinite (as a set containing a denumerable set), the statement will follow from Problem 4.

Since $A$ and $B$ are denumerable, there exist bijective functions $f_1 : \mathbb{N} \to A$, and $f_2 : \mathbb{N} \to B$. Let $d = |C|$ (it is a positive integer, since $C$ is finite), and let $C = \{c_1, \ldots, c_d\}$. Define the function $h : \mathbb{N} \to A \cup B \cup C$ by: $h(n) = c_m$ when $1 \leq n \leq d$; and $h(n) = f_1\left(\frac{n - d + 1}{2}\right)$ if $n - d$ is odd, and $h(n) = f_2\left(\frac{n - d}{2}\right)$ if $n - d$ is even, for $n > d$. Then $h$ is surjective, since for every $a \in A$, there exists $m \in \mathbb{N}$ such that $f_1(m) = a$; then $h(2m + d - 1) = a$; similarly, for every $b \in B$, there exists $q \in \mathbb{N}$ such that $f_2(q) = b$, and then $h(2q + d) = b$. (In fact, all we are doing here is listing the elements of $A \cup B \cup C$ in the following order: $c_1, \ldots, c_d, a_1, b_1, a_2, b_2, \ldots$)

Note that $h$ doesn’t have to be injective (there could be repetitions on this list if $A$, $B$ and $C$ have some elements in common), but it suffices to construct a surjective function.

6. Prove that if a set $A$ contains an uncountable subset, then $A$ is uncountable.

**Note:** We use this statement in the proof that the interval $(0,1)$ is uncountable.
Solution: Proof by contradiction: suppose $A$ was countable. Then either $A$ is finite or it is denumerable. Clearly, a finite set cannot contain an uncountable subset (all subsets of a finite set are finite). Then, $A$ is denumerable. But we proved in class that every subset of a denumerable set is either denumerable or finite. We get a contradiction with the assumption that $A$ contains an uncountable subset.

7. Let $A$ be any uncountable set, and let $B \subset A$ be a countable subset of $A$. Prove that $|A| = |A - B|$.

**Hint.** This is a generalization of one of the last problems from Workshop 5, and has a very similar solution.

**Solution:** By the previous problem, $A - B$ is infinite (if it was finite, $A$ would have been a union of a finite set and a countable set, and therefore, countable). Let $C = \{c_1, \ldots c_n, \ldots \}$ be any denumerable subset of $A - B$ (it seems obvious that we can just pick an element $c_1$ from the infinite set $A - B$, then pick any element $c_2$ from the set of remaining elements, and keep doing it; note, however, that technically, this is the axiom of choice, which is mentioned in 10.5; it’s not on the exam though). Once we have the set $C$, we know that $C$ and $C \cup B$ are both denumerable, and therefore there exists a bijective function $f : C \cup B \to C$. Now, define $h : A \to A - B$ by:

$$h(x) = \begin{cases} 
  x, & x \in A - (C \cup B) \\
  f(x) & x \in C \cup B.
\end{cases}$$

Then by the problem 2b) (see its solution), $h$ is a bijective function from $A$ to $A - B$.

8. (a) If $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of finite subsets of $\mathbb{N}$, show that $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is denumerable.

(b) If $\mathcal{P}_{\text{inf}}(\mathbb{N})$ denotes the set of infinite subsets of $\mathbb{N}$, show that $\mathcal{P}_{\text{inf}}(\mathbb{N})$ is uncountable.

**Hint:** Use the previous problem.

**Solution:** (a) Let $A_n$ denote the set of subsets of $\mathbb{N}$ that are contained in $\{1, \ldots, n\}$. Then $|A_n| = 2^n$ (we proved this by induction). Now, note that any finite subset of $\mathbb{N}$ is contained in $\{1, \ldots, n\}$ for a large enough number $n$. Therefore $\mathcal{P}_{\text{fin}}(\mathbb{N}) = \bigcup_{n=1}^{\infty} A_n$ is a denumerable union of finite sets and is clearly an infinite set. By Problem 2 on Workshop 5 this set is denumerable.

(b) Assume to the contrary that $B_1 = \mathcal{P}_{\text{inf}}(\mathbb{N})$ is countable. By (a) $B_2 = \mathcal{P}_{\text{fin}}(\mathbb{N})$ is denumerable. If $B_n = \emptyset$ for all $n \geq 3$ then $\mathcal{P}(\mathbb{N}) = \bigcup_{n=1}^{\infty} B_n$ is a denumerable union of countable sets and so by Q3 on Workshop 5 is also denumerable (it is trivially infinite so that result applies). This contradicts the fact (proven in class) that $\mathcal{P}(\mathbb{N})$ is uncountable. Therefore the set of infinite subsets of $\mathbb{N}$ must be uncountable.

9. Let $A, B$ be sets. Prove that
if \(|A - B| = |B - A|\) then \(|A| = |B|\).

Hint: draw a careful picture.

**Solution:** Given a bijection \(f : (A - B) \to (B - A)\) define

\[
g : A \to B \\
g(x) = \begin{cases} 
  f(x) & x \in (A - B) \\
  x & x \not\in (A - B)
\end{cases}
\]

![Diagram of sets A, B, A \cap B, and function f](image)

- Let \(g : A \to B\) be defined as above. We need to show that \(g\) is injective and surjective.

- **Injective.** Let \(x, z \in A\) and assume \(g(x) = g(z)\). This image must be in \(B\), but it may either be in \(A\) or not in \(A\) (that is, either \(y \in A \cap B\) or \(y \in B - A\)).
  
    - Assume \(g(x) = g(z) \not\in A\). Then both \(x, z \in A - B\) (otherwise their images under \(g\) would be in \(A\)). Hence \(g(x) = f(x)\) and \(g(z) = f(z)\). Since \(f\) is injective, it follows that \(x = z\).
    
    - Now assume that \(g(x) = g(z) \in A\). Then both \(x, z \in A\) (otherwise their images under \(g\) would be in \(B - A\)). Then \(g(x) = x\) and \(g(z) = z\) and so \(x = z\).

Hence \(g\) is injective.

- **Surjective.** Let \(y \in B\). Either \(y \in A\) or \(y \not\in A\) (that is, either \(y \in A \cap B\) or \(y \in B - A\)).
  
    - Assume \(y \in A\) then let \(x = y\). By the definition of \(g\), \(g(x) = x = y\).
    
    - Now assume \(y \not\in A\), then since \(f\) is surjective, there exists \(x \in A - B\) so that \(f(x) = y\). Now since \(x \in A - B\), it follows that \(g(x) = f(x) = y\).

Hence \(g\) is surjective.

- Hence \(g(x)\) is bijective as required.

10. Let \(\{0, 1\}^\mathbb{N}\) be the set of all possible sequences of 0s and 1s. Corollary 10.22 in the text states that in fact, the cardinality of this set is continuum: \(|\mathbb{R}| = |\{0, 1\}^\mathbb{N}|\) (see also Problem 5 on Workshop 6). Using this fact, prove that \(|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|\).
Solution: Define $f : \{0,1\}^\mathbb{N} \times \{0,1\}^\mathbb{N} \to \{0,1\}^\mathbb{N}$ by

$$f(\{x_n\}, \{y_n\})_m = \begin{cases} x_{(m+1)/2} & \text{if } m \text{ is odd} \\ y_{m/2} & \text{if } m \text{ is even.} \end{cases}$$

We claim $f$ is bijective. Let $\{z_n\} \in \{0,1\}^\mathbb{N}$. Define $x_n = z_{2n-1}$ and $y_n = z_{2n}$. Then the above definitions easily imply that $\{z_n\} = f(\{x_n\}, \{y_n\})$. Therefore $f$ is onto. Assume $f(\{x_n\}, \{y_n\}) = f(\{x'_n\}, \{y'_n\})$. Equating the even coefficients of these two sequences we find that $\{x_n\} = \{x'_n\}$ and equating the odd coefficients we get $\{y_n\} = \{y'_n\}$. Therefore $(\{x_n\}, \{y_n\}) = (\{x'_n\}, \{y'_n\})$, and so $f$ is one-to-one.

It follows from the above that $|\{0,1\}^\mathbb{N} \times \{0,1\}^\mathbb{N}| = |\{0,1\}^\mathbb{N}|$. Now use Question 2(a) and Corollary 10.22, the above equality and finally Corollary 10.22 again to conclude that

$$|\mathbb{R} \times \mathbb{R}| = |\{0,1\}^\mathbb{N} \times \{0,1\}^\mathbb{N}| = |\{0,1\}^\mathbb{N}| = |\mathbb{R}|.$$ 

This completes the proof.