1. **Question 6.2.** Prove that any non-empty subset of a well-ordered set of real numbers is well-ordered.

2. Prove that for all \( n \in \mathbb{N} \)

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.
\]

3. Let \( a, b \in \mathbb{Z} \) and \( m \). Using mathematical induction, prove that if \( a \equiv b \mod m \) then \( a^n \equiv b^n \) for all \( n \in \mathbb{N} \).

4. Let \( A \) be a finite set, and let \( \mathcal{P}(A) \) be the set of all its subsets. Prove the fact that \( |\mathcal{P}(A)| = 2^{|A|} \) by mathematical induction.

5. Use induction to prove DeMorgan’s law for \( n \) sets \( (n \geq 1) \): let \( A_1, \ldots, A_n \) be sets; prove that

\[
\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.
\]

6. Prove that for all natural numbers \( n \geq 10 \), \( 2^n > n^3 \).

7. Let \( F_1, F_2, \ldots, F_n, \ldots \), be the sequence of Fibonacci numbers: by definition, \( F_1 = F_2 = 1 \), and the sequence is defined *recursively* by the formula \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 3 \). (Thus, we get the sequence 1, 1, 2, 3, 5, 8, 13, 21, \ldots ) Prove the formula

\[
F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).
\]

Two remarks: (1) Note that it’s not even completely obvious why the expression on the right is a rational number! But you will prove that it is, in fact, an integer, namely, the \( n \)-th Fibonacci number. (2) The number \( \varphi = \frac{1+\sqrt{5}}{2} \) is called the *golden ratio*.

8. A graph is called *complete* if any two vertices are connected by an edge. Prove that the complete graph with \( n \) vertices contains precisely \( n(n-1)/2 \) edges.

9. A graph is called a *tree* if it is connected and contains no cycles (equivalently, there is precisely one path from any vertex \( v_1 \) to any other vertex \( v_2 \)). Prove that:

   (a) Any tree must contain at least one vertex of degree 1 (that is, a vertex with only one edge leading to it; sometimes such a vertex in a tree called “a leaf”).

   *Hint: consider the longest possible path in this graph, and think about its ends.*

   (b) Prove that any tree with \( n \) vertices contains precisely \( n - 1 \) edges.