

1. Prove that the number 123456782 cannot be represented as $a^2 + 3b^2$ for any integers a and b . (*Hint: Consider the remainder mod 3*).
2. (a) Prove that there are infinitely many primes p such that $p \equiv 3 \pmod{4}$. *Hint: try to proceed the same way as in Euclid's proof of the statement that there are infinitely many prime numbers; but instead of making the number $N = p_1 \dots p_n + 1$, make a number N that is definitely congruent to 3 modulo 4 (and that still differs by 1 from a number that is divisible by all of p_1, \dots, p_k).*
(b) Could this proof have worked for the primes congruent to 1 modulo 4?
3. Find the last digit of the number 2016^{2016} .
4. Prove that there do not exist integers a , b and c such that

$$12345678910111213 = a^2 + 25b^2 + 5c^2.$$

5. Suppose that the following three statements are true:
 1. Rainbows are colourful.
 2. If it isn't sparkly, then it must be extravagant.
 3. Colourful things are never extravagant.

What can you conclude about rainbows?

6. Consider the statement (implication):

If Bill takes Sam to the concert, then Sam will take Bill to dinner.

Which of the following implies that this statement is true:

- (a) Sam takes Bill to dinner only if Bill takes Sam to the concert.
 - (b) Either Bill doesn't take Sam to the concert or Sam takes Bill to dinner.
 - (c) Bill takes Sam to the concert.
 - (d) Bill takes Sam to the concert and Sam takes Bill to dinner.
 - (e) Bill takes Sam to the concert and Sam doesn't take Bill to dinner.
 - (f) The concert is canceled.
 - (g) Sam doesn't attend the concert.
7. Consider the statement: *The fish are biting and there are no bugs, or the fish are not biting and there are bugs, or it is winter.* Write out the negation of the above in English. You should simplify your answer as much as possible, being sure of course that it is logically equivalent to the negation. Justify your answer.
 8. Let $P(x)$ and $Q(x)$ be open sentences where the domain of the variable x is a set S . Which of the following implies that $\sim P(x) \Rightarrow Q(x)$ is false for some $x \in S$?
 - (a) $P(x) \wedge Q(x)$ is false for all $x \in S$.
 - (b) $P(x)$ is true for all $x \in S$.

- (c) $Q(x)$ is true for all $x \in S$.
- (d) $P(x) \vee Q(x)$ is false for all $x \in S$.

9. Suppose you have the following information about the population of the planet QE220:

- Among the inhabitants of QE220 who can watch TV, not all have antennae on their head.
- The inhabitants of QE220 that are green and do not have antennae, cannot watch TV.

Does it follow that not all the inhabitants of QE220 that can watch TV are green? Justify your answer.

Hint. Start by writing down the statement you are asked about.