Some harder problems on cardinality.

These are two series of problems with specific goals: the first goal is to prove that the cardinality of the set of irrational numbers is *continuum*, and the second is to prove that the cardinality of $\mathbb{R} \times \mathbb{R}$ is continuum, without using Cantor-Bernstein-Schröeder Theorem.

1. Does there exist a continuous bijective function $f : \mathbb{R} \to \mathbb{R} - \{1\}$? Explain.
   Hint: Recall the Intermediate Value Theorem.

2. Prove that $|(0, 1)| = |[0, 1)|$.
   Hint. Do not try to write a formula for a bijective function. Instead, choose an infinite countable subset A of $(0, 1)$. Then make a bijection between A and $A \cup \{0\}$. Then define your function on the rest of the interval. The resulting function will be very discontinuous, but that does not matter!

3. Let A be any uncountable set, and let $B \subset A$ be a countable subset of A. Prove that $|A| = |A - B|$.
   Hint. This is a generalization of the previous problem, and has a very similar solution.

4. (a) Prove that if A is a countable subset of real numbers, then $|\mathbb{R} - A| = |\mathbb{R}|$.
   Hint. Modify the previous hint as follows. Choose $b_n \in (n, n + 1) - A$ for all $n \in \mathbb{N}$. (Why does such a b_n exist?) Hence $B = \{b_n : n \in \mathbb{N}\}$ is a countable set of reals disjoint from A. Now choose a bijection $f : A \cup B \to B$. (Explain why this exists using an earlier question.) Finally extend f to a bijection from $\mathbb{R}$ to $\mathbb{R} - A$.

   (b) Prove that $|\mathbb{I}| = |\mathbb{R}|$, where $\mathbb{I}$ is the set of irrationals as usual.

5. Let $A = \{0, 1\}^{\mathbb{N}}$ be the set of all possible sequences of 0's and 1's. Prove that A is uncountable.

6. Let $A = \{0, 1\}^{\mathbb{N}}$ be the set of all possible sequences of 0's and 1's, as in the previous problem. Prove that $|A \times A| = |A|$.

7. (a) Prove that the cardinality of the set from the previous problem is, in fact, continuum.
   Hint: use binary representation of real numbers.

   (b) Prove that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.

   (c) What is wrong with the following sketch of a proof that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$:
   let $\mathbb{R} = \{a_1, \dots, a_n, \dots\}$*. Then* $\mathbb{R} \times \mathbb{R}$ *is the set of pairs* (a_i, b_j) *where* $i, j \in \mathbb{N}$*, and now we can arrange these pairs in a table, and use the algorithm for traversing the table to put them into one long list* $\{c_1, \dots c_n, \dots\}$*.*
   (Note that this proof works fine for $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q} \times \mathbb{Q}$).

   Definition. Let A be an arbitrary set and let B be a subset of A. Define the function $\chi_B : A \to \{0, 1\}$ by the formula

$$\chi_B(x) := \begin{cases} 0 \text{ if } x \notin B, \\ 1 \text{ if } x \in B. \end{cases}$$

   This function is called the *characteristic function* of B.

8. Let $A = \{0,1\}^{\mathbb{N}}$ be the set of all possible sequences of 0's and 1's. Find a very natural bijective function between A and $\mathcal{P}(\mathbb{N})$.
(**Hint:** construct a function from $\mathcal{P}(\mathbb{N})$ to P, whose definition uses the notion of the characteristic function).

9. (a) If $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of finite subsets of $\mathbb{N}$, show that $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is countable.

   (b) If $\mathcal{P}_{\text{inf}}(\mathbb{N})$ denotes the set of infinite subsets of $\mathbb{N}$, show that $\mathcal{P}_{\text{inf}}(\mathbb{N})$ is uncountable.

   Hint: Use the previous problem.