Some harder problems on cardinality.

These are two series of problems with specific goals: the first goal is to prove that the cardinality of the set of irrational numbers is continuum, and the second is to prove that the cardinality of $\mathbb{R} \times \mathbb{R}$ is continuum, without using Cantor-Bernstein-Schröeder Theorem.

1. Does there exist a continuous bijective function $f : \mathbb{R} \to \mathbb{R} - \{1\}$? Explain.
   
   Hint: Recall the Intermediate Value Theorem.

2. Prove that $|(0, 1)| = |[0, 1)|$.
   
   Hint. Do not try to write a formula for a bijective function. Instead, choose an infinite countable subset $A$ of $(0, 1)$. Then make a bijection between $A$ and $A \cup \{0\}$. Then define your function on the rest of the interval. The resulting function will be very discontinuous, but that does not matter!

3. Let $A$ be any uncountable set, and let $B \subset A$ be a countable subset of $A$. Prove that $|A| = |A - B|$.
   
   Hint. This is a generalization of the previous problem, and has a very similar solution.

4. (a) Prove that if $A$ is a countable subset of real numbers, then $|\mathbb{R} - A| = |\mathbb{R}|$.
   
   Hint. Modify the previous hint as follows. Choose $b_n \in (n, n + 1) - A$ for all $n \in \mathbb{N}$. (Why does such a $b_n$ exist?) Hence $B = \{b_n : n \in \mathbb{N}\}$ is a countable set of reals disjoint from $A$. Now choose a bijection $f : A \cup B \to B$. (Explain why this exists using an earlier question.) Finally extend $f$ to a bijection from $\mathbb{R}$ to $\mathbb{R} - A$.

   (b) Prove that $|\mathbb{I}| = |\mathbb{R}|$, where $\mathbb{I}$ is the set of irrationals as usual.

5. Let $A = \{0, 1\}^\mathbb{N}$ be the set of all possible sequences of 0’s and 1’s. Prove that $A$ is uncountable.

6. Let $A = \{0, 1\}^\mathbb{N}$ be the set of all possible sequences of 0’s and 1’s, as in the previous problem. Prove that $|A \times A| = |A|$.

7. (a) Prove that the cardinality of the set from the previous problem is, in fact, continuum.
   
   Hint: use binary representation of real numbers.

   (b) Prove that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.

   (c) What is wrong with the following sketch of a proof that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$:
   
   let $\mathbb{R} = \{a_1, \ldots, a_n, \ldots\}$. Then $\mathbb{R} \times \mathbb{R}$ is the set of pairs $(a_i, b_j)$ where $i, j \in \mathbb{N}$, and now we can arrange these pairs in a table, and use the algorithm for traversing the table to put them into one long list $\{c_1, \ldots, c_n, \ldots\}$.

   (Note that this proof works fine for $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q} \times \mathbb{Q}$).

Definition. Let $A$ be an arbitrary set and let $B$ be a subset of $A$. Define the function $\chi_B : A \to \{0, 1\}$ by the formula

$$\chi_B(x) := \begin{cases} 
0 \text{ if } x \notin B, \\
1 \text{ if } x \in B.
\end{cases}$$

This function is called the characteristic function of $B$. 
8. Let \( A = \{0, 1\}^\mathbb{N} \) be the set of all possible sequences of 0’s and 1’s. Find a very natural bijective function between \( A \) and \( \mathcal{P}(\mathbb{N}) \).

\( \text{(Hint: construct a function from } \mathcal{P}(\mathbb{N}) \text{ to } \mathcal{P}, \text{ whose definition uses the notion of the characteristic function).} \)

9. (a) If \( \mathcal{P}_{\text{fin}}(\mathbb{N}) \) denotes the set of finite subsets of \( \mathbb{N} \), show that \( \mathcal{P}_{\text{fin}}(\mathbb{N}) \) is countable.

(b) If \( \mathcal{P}_{\text{inf}}(\mathbb{N}) \) denotes the set of infinite subsets of \( \mathbb{N} \), show that \( \mathcal{P}_{\text{inf}}(\mathbb{N}) \) is uncountable.

\( \text{Hint: Use the previous problem.} \)