Topics

- Basic logic (negate an implication)
- Quantifier
- Indexed collections
  - Product $A \times B$
  - De Morgan
  - Set operation

Congruences

- Induction
- Rationality

Functions

- Cardinality
- Graphs
  - Countable
  - Uncountable

Format: (Similar as previous year) Similar as old finals.

Ignoring sequence / limits

1 graph question (easier than 2nd MT)

Be honest with your proof work on exam.

Negation an implication:

$\neg (P \implies Q) \equiv P \land \neg Q$ (Not an Implication)

$P \implies Q \equiv \neg P \lor Q$

Indexed Collections

(And infinite sums)

Let $I = \{ n^2 \mid n \in \mathbb{Z} \}$

Describe $\bigcap_{i \in I} S_i$, where $S_i = (i-1, i+1)$; $\bigcup_{i \in I} S_i$

Write $\sum_{i \in I}$ as something we can understand.

$I = \{0, 1, 4, 9, 16, 25, \ldots \}$ - Squares of integers.

Always write down set $I$ in a form that you feel comfortable with.

\[
S_2 = (0-1, 0+1) = (-1, 1)
\]

Coming from set $I$

\[
P_1 = (1-1, 1+1) = (0, 2)
\]

\[
P_2 = (2-1, 2+1) = (1, 3)
\]

\[
(1, 1) \land (0, 2) \land (3, 5) \land \ldots = \emptyset
\]
\[ S_1 = (-1, 1) \cup (0, 2) \cup (3, 5) \cup \ldots \]
\[ = \cup_{n=0}^{\infty} (n^2-1, n^2+1) \]

plugged in the definition of the set \( I \)

\[ \sum_{n \in I} n = 0 + 1 + 4 + 9 + 6 + \ldots \]
\[ = \sum_{n=0}^{\infty} n^2 \]

**What is the complement?**

\[ \bigcup_{n \in I} S_n = \bigcap_{n \in I} \overline{S_n} \]
\[ = \bigcap_{n=0}^{\infty} (-\infty, n^2-1] \cup [n^2+1, \infty) \]

\[ \overline{S_n^2} \]

\[ \text{For } x \in [0, 1], \text{ let } S_x = \{ (x, y) : y \in [0, 1] \} \]

\[ \text{Describe } \bigcup_{x \in [0, 1]} S_x = [0, 1] \times [0, 1] \]

\[ \text{Uncountable} \]

**Formal Proof:**

\[ \bigcup_{x \in [0, 1]} S_x \subset [0, 1] \times [0, 1] \]

1) Let \( a \in \bigcup_{x \in [0, 1]} S_x \), then \( a \in S_x \) for some \( x \in [0, 1] \)

Then \( a = (x, y) \) for some \( y \in [0, 1] \)

Then \( (x, y) \in [0, 1] \times [0, 1] \) by definition of product.

2) Converse: Let \( (x, y) \in [0, 1] \times [0, 1] \),
Then \((x_0, y_0) \in S_{x_0}\), so \(x_0 \in U\) \(S_x\).

**Different interpretation:**

What if \((x, y)\) as an interval? \(\rightarrow\) ask your clarifications
was not intended; cannot be fixed variable.

**What if we write:**

\[ S_x = \{(x, y) : x \in [0, 1], y \in [0, 1]\} \]

**NOT OK**

\[ S = \{(x, y) : x \in [0, 1], y \in [0, 1]\} \]

\[ = [0, 1] \times [0, 1] \]

\(-\) it's fine!

**Cardinality**

*Never write* \(U\) \(A_x = A_1 \cup A_2 \cup A_3 \cup \cdots \)

the interval \([0, 1]\) is uncountable.

its points cannot be listed as \(\{x_0, x_2, \cdots\}\)

**Def.:** \(|A_1| = |B_1|\) means:
there exists a bijective function \(f: A \rightarrow B\).

(For finite sets, to establish \(|A_1| = |B_1|\), can just count the number of elements.)

"Smaller cardinality" works differently for infinite vs. finite sets.

For finite sets:

If \(\exists f: A \rightarrow B\) injective but not surjective,
then \(|A| < |B|\).

For infinite sets, this is false: always false!

Example: \(f: \mathbb{N} \rightarrow \mathbb{N}\), \(f(n) = n + 1\)

Injective: (one-to-one): if \(n_1 \neq n_2\) then \(f(n_1) \neq f(n_2)\)

Not surjective: \(f^{-1}(\{1, 3\}) = \emptyset\)

Is \(\mathbb{N} \times \mathbb{N}\) larger cardinality than \(\mathbb{N}\)? \((\text{NO})\) the same.
\[ \mathbb{N} \times \mathbb{N} = \{(a, b) : a, b \in \mathbb{N}\} \]

- works for countable sets

A related example:

\[ \bigcup_{n=1}^{\infty} A_n \]

- An is denumerable for all n

\[ A_n = \{a_1^{(n)}, a_2^{(n)}, \ldots, a_k^{(n)} \ldots \} \]

- Fact: \(|\mathbb{R}| > |\mathbb{N}|\)

- Q: What set has cardinality bigger than \(\mathbb{R}\)?

  Answer: \(\mathcal{P}(\mathbb{R}) = \) the set of all subsets of \(\mathbb{R}\)

- Fact: \(|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|\) (won't test proof of this statement in this final)

What is \(|\mathcal{P}(\mathbb{N})|\)?

\(\mathcal{P}(\mathbb{N}) \rightarrow \) set of sequence of 0, 1

\[ 010110 \ldots \]

\(f(A) = \) the sequence \(S_A\), where

\[ S_A = \{1, 0 : \text{if } 2k \in A \} \]

Ex: \(A = \{1, 2, 5\} \rightarrow \{010100 \ldots \} \)

Why is this function injective?

Let \(A_1, A_2\) be different subsets of \(\mathbb{N}\)

To prove: they correspond to different sequences.

Since \(A_1, A_2\) are not the same set

exists \(n \in A_1, \) but not in \(A_2\) (without loss of generality)
or exists \( m \in A_2 \) s.t. \( m \notin A_1 \)

Then, in the sequence \( f(A_1) \) the \( n \)th term is 1 and in \( f(A_2) \) the \( n \)th term is 0.

Why is surjective?

let \( s = s_1 s_2 s_3 s_4 \ldots \) be a sequence of 0's and 1's

Need to prove: \( \exists \ A \ St.: f(A) = s, s_2 s_3 \ldots \)

Let \( A = \{ \exists s \in \mathbb{N} : s_1 = 1 \} \)

We proved:

\[
|P(\mathbb{N})| = \text{set of sequences of } \{0, 1\} \\
= |\mathbb{N}| \\
(\text{every real number has binary representation.})
\]

- Methods for working with \( |\mathbb{R}| \):
  1. \( |\mathbb{R}| = \text{any interval} \) use function like \( \tan(x) \)

      \[
      \tan(x) : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}
      \]

      Use linear functions to switch between different intervals.

  2. USE (without proof is ok) that:

      \[
      |\mathbb{R}| = \{\text{sequence of } 0, 1 \} \\
      = \{\text{sequence of } 0, 1, \ldots 9 \} \\
      \]

      (works for \( |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}| \) decimal expansions)

  3. \( |\text{Uncountable Set} - \text{Countable set}| = |A| \)

      \[
      |A - B| = |A| \\
      A: \text{any Uncountable set} \\
      B: \text{countable set.}
      \]
Function:

- bijective function between two open intervals, use linear
  \[ f : (a, b) \rightarrow (c, d) \]
  \[ y = mx + k \]
  find \( m, k \).

- from open interval to closed interval
  \[ (a, b) \rightarrow [c, d] \]
  use (3)

#8 from Dec 2013

\[ f : (1, 1) \rightarrow \mathbb{R} \]

\[ f(x) = \frac{1}{x} - x \]

Prove it is a bijective function and then find \( f^{-1} \).

1) injective: want to prove if \( x_1 \neq x_2 \) then

\[ \frac{1}{x_1} - x_1 \neq \frac{1}{x_2} - x_2 \]

Contrapositive: Suppose \( \frac{1}{x_1} - x_1 = \frac{1}{x_2} - x_2 \)

Given that \( x_1, x_2 \neq 0 \)

\[ x_2(1 - x_1^2) = (1 - x_2^2)x_1 \]

\[ x_2 - x_1 + x_2 x_1 (x_2 - x_1) = 0 \]

\[ (x_2 - x_1) (x_1 + x_2) = 0 \]

\[ x_1 + x_2 \neq 0 \]

\[ x_2 - x_1 = 0 \]

\[ x_1 = x_2 \]

Try: maybe \( f \) is decreasing?

\[ f'(x) = -\frac{1}{x^2} - 1 < 0 \]

Also, could prove algebraically: \( x_2 > x_1 \)

\[ \Rightarrow f(x_2) < f(x_1) \]
Surjective? Let \( y \in \mathbb{R} \). Need to prove \( \exists x \in (-1, 150^\circ) \) such that \( \frac{1}{x} - x = y \). Solve for \( x \). Show your \( x \) lies in \((-1, 150^\circ)\).

**Alternative:**
\[
\begin{align*}
\lim_{x \to 0^-} f(x) &= -\infty \\
\lim_{x \to 0^+} f(x) &= +\infty \\
\lim_{x \to 1} f(x) &= 0
\end{align*}
\]
Use IVT.

For \( f^{-1}(y) \):
we wanted \( x \) s.t. \( f(x) = y \)
\[
x = \frac{-y}{2} \pm \sqrt{\frac{y^2}{4} + 1}
\]
Recall: Need \( x \in (-1, 150^\circ) \)
when \( y > 0 \), then \(-\sqrt{\frac{y^2}{4} + 1} < -1 \)
when \( y < 0 \) -----
should be only one should work.

- Congruence.
- If \( a \equiv 1 \pmod{2} \), \( b \equiv 3 \pmod{4} \)
  then \( a^2 + b = 0 \pmod{4} \)

**Lemma:** \( a^2 \equiv 1 \pmod{2} \), then \( a^2 \equiv 1 \pmod{4} \)

2 ways:
1. \( a = 2k + 1 \quad k \in \mathbb{Z} \quad b \equiv 3 \pmod{4} \).
    Then \( a^2 = (2k + 1)^2 = 4k^2 + 4k + 1 \equiv 1 \pmod{4} \).

2. \( a \equiv 1 \pmod{2} \)
   then \( a \equiv 1 \pmod{2} \) or \( a^2 \equiv 1 \quad \) or \( a^2 \equiv 9 \equiv 1 \pmod{4} \).

Now: \( a^2 \equiv 1 \pmod{4} \)
\( +1 \) \( b \equiv 3 \pmod{4} \)
\[ a^2 + b \equiv 1 + 3 \mod 4 \equiv 0 \mod 4 \]

- **Question (Dec. 2011)**
  - If \( p \) is prime, \( p > 3 \)
  - Then, \( p^2 \equiv 1 \mod 6 \)
  - \( p^2 \equiv 1 \)
  - or \( 25 \equiv 1 \mod 6 \)

*Start with:*

Since \( p \) is prime,
\[ p \not\equiv 0, 2, 4, 3 \mod 6 \]

**Lemma**: proof:
- \( p \equiv 0 \mod 6 \) obvious.
- Since \( p \) is odd, \( p \) cannot be \( \equiv 2 \mod 6 \).
  - \( 6k + 2, 6k + 4 \) are even.
- If \( p = 6k + 3 \),
  - then \( p \) is divisible by 3.
  - It can't be \( p \) prime.

**Graphs**

- **Connected**
  - Graph with degrees of vertices \( 1, 2, 3, 4, 4, 8 \)
    - does not exist
  - \( 1, 2, 3, 4, 4, 3, 3 \)
  - \( \# \) edges: \( \frac{1}{2} \times (1 + 2 + 3 + 4 + 4 + 3 + 3) \)

- In any group of people shaking hands,
  - there are two who shook the same number of hands.
(Any graph has to have 2 vertices of the same degree)

Let \( a \) be the number of vertices,
- Possible degrees: \( 0, 1, 2, \ldots, a-1 \)
- \( \frac{1}{a} \) cannot both occur!