Topics

- basic logic (*negate an implication)
- quantifier
- indexed collections
  product $A \times B$
- DeMorgan
  set operation

- congruences
- induction
- rationality

- functions
- cardinality
- graphs
  (Countable/uncountable)

format: (Similar as previous year) similar as old finals.
ignor sequence | limits
1 graph question (easier than 2nd MT)
be honest with your proof work on exam.

Negation an implication:

$\sim (P \Rightarrow Q) \equiv P \wedge \sim Q$ (Not an implication)

$P \Rightarrow Q \equiv \sim P \vee Q$

- Indexed Collections
  (and infinit sums)

- Let $I = \{ n^2 \mid n \in \mathbb{Z} \}$

  Describe $\bigcap_{i \in I} S_i$, where $S_i = (i-1, i+1)$; $\bigcup_{i \in I} S_i$

  Write $\sum_{i \in I} i$ as something we can understand.

  $I = \{0, 1, 4, 9, 16, 25, \cdots\}$ - squares of integers.

  always write down set I in a form that you feel comfortable with.

  $S_0 = (0-1, 0+1) = (-1, 1)$

  Coming from set I ← plug it into the def. of S_i

  $S_1 = (1-1, 1+1) = (0, 2)$

  $S_4 = (4-1, 4+1) = (3, 5)$

  $\underbrace{(-1, 1) \cap (0, 2) \cap (3, 5)}_{\phi} \cap \cdots = \phi$

$$\bigcup_{i=1}^{\infty} S_i = (-1,1) \cup (0,2) \cup (3,5) \cup \cdots$$

$$= \bigcup_{n=0}^{\infty} (n^2-1,\ n^2+1)$$

↑ plugged in the definition of the set I

$$\sum_{i\in I} i = 0+1+4+9+16+\cdots$$

$$= \sum_{n=0}^{\infty} n^2$$

What is the complement?

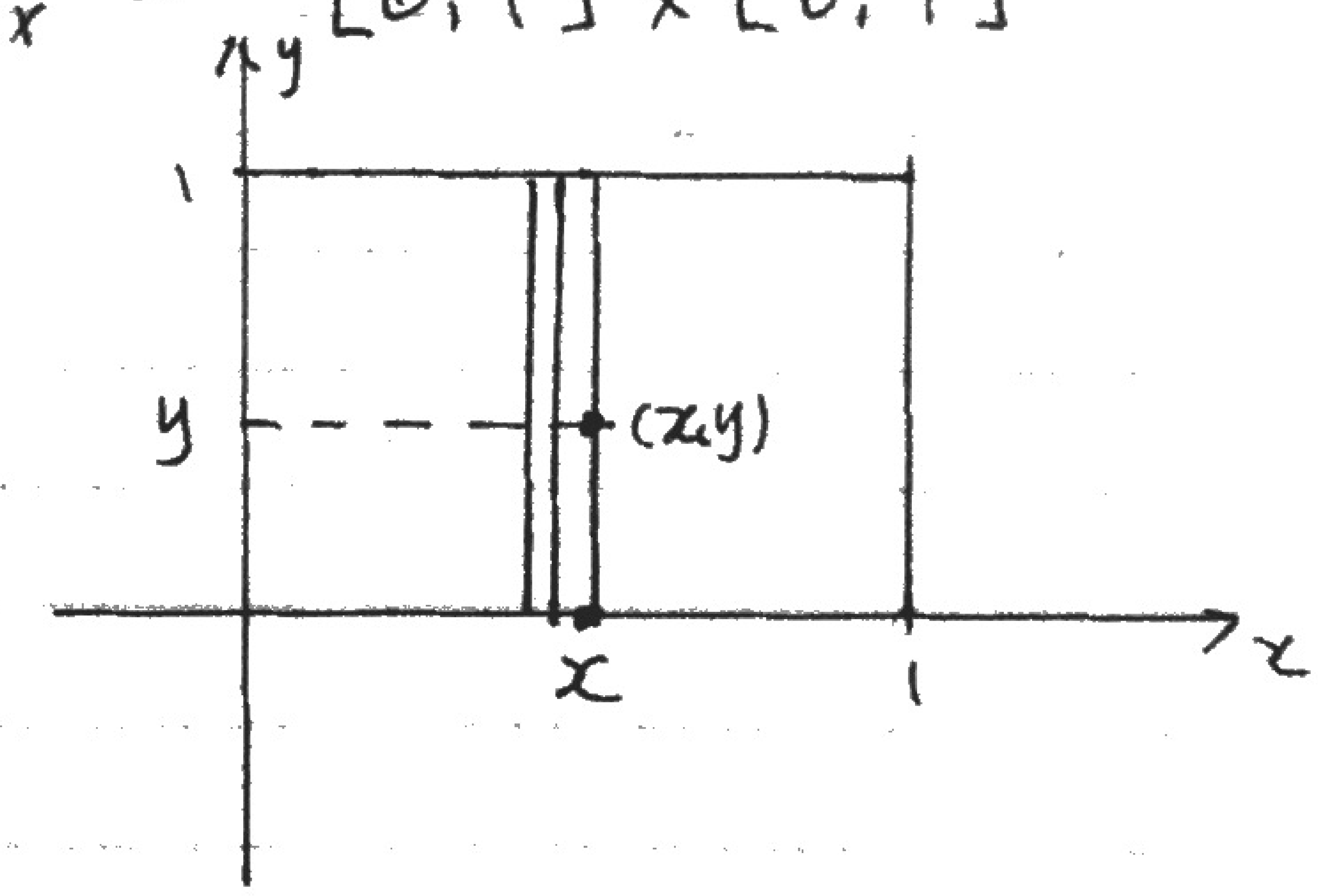
$$\overline{\bigcup_{i\in I} S_i} = \bigcap_{i\in I} \overline{S_i}$$

$$= \bigcap_{n=0}^{\infty} \underbrace{(-\infty, n^2-1] \cup [n^2+1, \infty)}_{\overline{S_{n^2}}}$$

- For $x \in [0,1]$, let $S_x = \{(x,y) : y \in [0,1]\}$

Describe $\bigcup_{x\in[0,1]} S_x = [0,1] \times [0,1]$

$\boxed{\text{uncountable}}$



Formal proof:

$$\bigcup_{x\in[0,1]} S_x \subseteq [0,1]\times[0,1]$$

1) Let $a \in \bigcup_{x\in[0,1]} S_x$, then $a \in S_x$ for some $x \in [0,1]$

Then $a = (x,y)$ for some $y \in [0,1]$

Then $(x,y) \in [0,1] \times [0,1]$ by definition of product.

2) Converse: Let $(x_0,y) \in [0,1] \times [0,1]$,

Then $(x_0, y_0) \in S_{x_0}$, so it belongs to $\bigcup_{x \in [0,1]} S_x$.

Different interpretation:

what if (x, y) as an interval? → ask your clarification in exam if you have such a question.
was not intended!

what if we write: $S_x = \{(x, y) : x \in [0,1], y \in [0,1]\}$ (fixed; variable; cannot be ⊗)

NOT OK

$S = \{(x,y) : x \in [0,1], y \in [0,1]\}$
$= [0,1] \times [0,1]$
— it's fine!

- Cardinality

  - Never write $\bigcup_{x \in [0,1]} A_x = A_1 \cup A_2 \cup A_3 \cup \cdots$

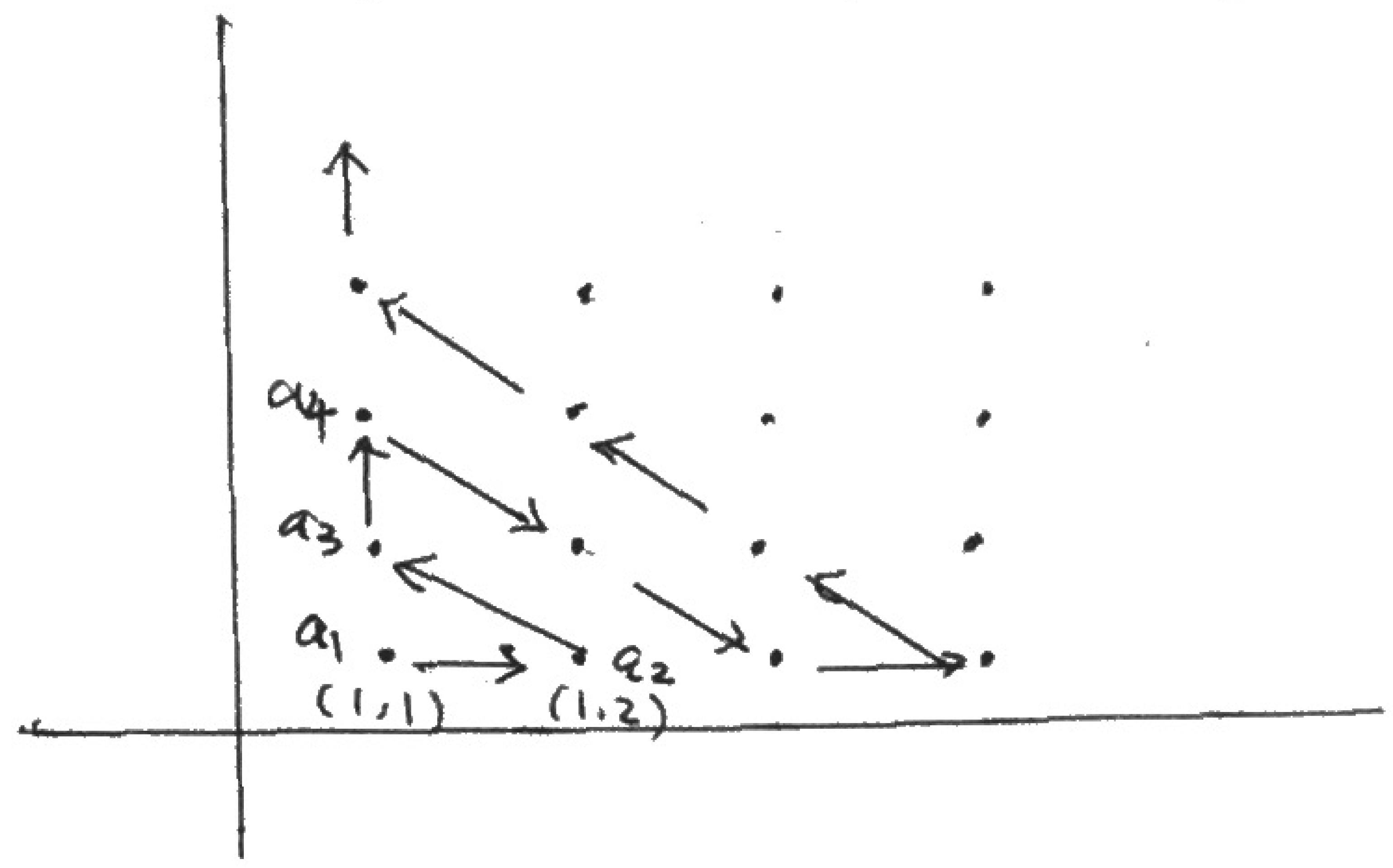
    the interval $[0,1]$ is uncountable.
    Its points cannot be listed as $\{x_1, x_2, \cdots\}$

  - Def: $|A| = |B|$ means:
    there exists a bijective function $f: A \to B$.
    (for finite sets, to establish $|A| = |B|$, can just count the number of element).

    "Smaller cardinality" works differently for infinit vs. finite sets.
    - for finite sets:
      If $\exists f: A \to B$ injective but not surjective,
      then $|A| < |B|$
    - for infinite sets, this is false! always false!
      example: $f: \mathbb{N} \to \mathbb{N}$ $f(n) = n+1$
      injective: (one-to-one): if $n_1 \neq n_2$ then $f(n_1) \neq f(n_2)$
      Not surjective: $f^{-1}(\{1\}) = \phi$

- Is $\mathbb{N} \times \mathbb{N}$ larger cardinality than $\mathbb{N}$? (NO) the same.

$\mathbb{N} \times \mathbb{N} = \{ (a,b) : a, b \in \mathbb{N} \}$



← works for countable sets

A related example:

$\bigcup_{n=1}^{\infty} A_n$, A_n is denumerable for all n

$A_n = \{ a_1^{(n)}, a_2^{(n)}, \cdots, a_k^{(n)} \cdots \}$

- Fact : $|\mathbb{R}| > |\mathbb{N}|$

  Q: What set has cardinality bigger than $\mathbb{R}$?

  Answer: $P(\mathbb{R})$ = the set of all subsets of $\mathbb{R}$

- Fact : $|P(A)| > |A|$ (won't test proof of this statement in this final).

What is $|P(\mathbb{N})|$?

$P(\mathbb{N}) \to$ set of sequence of 0, 1

010110 ⋯

$f(A)$ = the sequence S_i, where

$$S_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

Ex: $A = \{1, 2, 5\} \longmapsto 11001000 \cdots$

$\downarrow \downarrow \quad \uparrow$
$i=1 \; i=2 \quad i=5$

Why is this function injective?

Let A_1, A_2 be different subsets of $\mathbb{N}$

To prove: they correspond to different sequences.

Since A_1, A_2 are not the same set, exists $n \in A_1$, but not in A_2 (without loss of generality)

| or exists $m \in A_2$ s.t. $m \notin A_1$ |

Then, in the sequence $f(A_1)$ the n^{th} term is 1 and in $f(A_2)$ the n^{th} term is 0

Why is surjective?

let $s = s_1 s_2 s_3 s_4 \cdots$ be a sequence of 0's and 1's

Need to prove: $\exists A$ s.t.: $f(A) = s_1 s_2 s_3 \cdots$

Let $A = \{ i \in \mathbb{N} : s_i = 1 \}$

We proved:

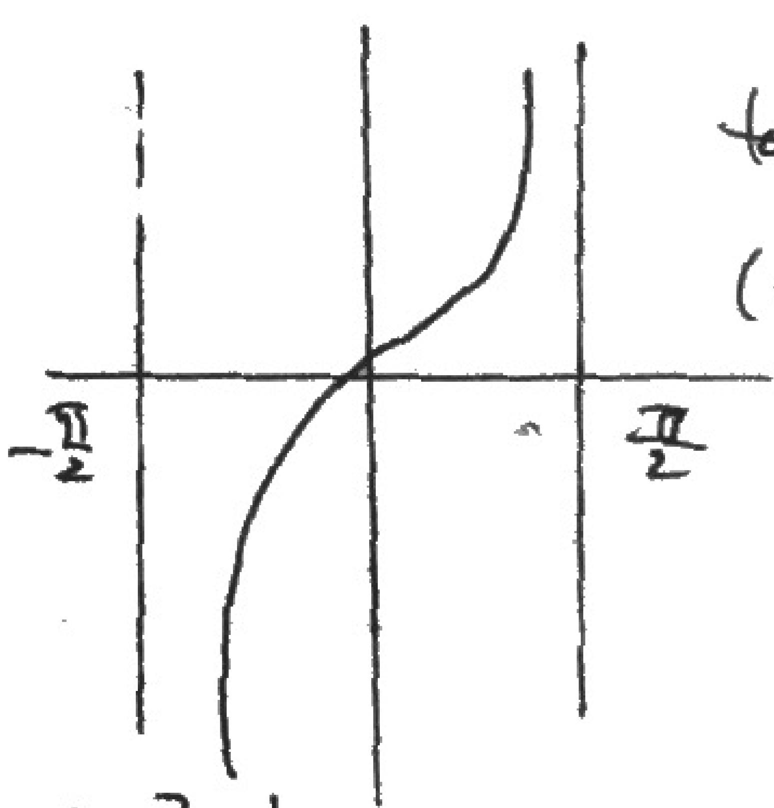
$$|P(\mathbb{N})| = |\text{set of sequences of } \{0, 1\}|$$
$$= |\mathbb{R}|$$

(every real number has binary representation.)

• Methods for working with $|\mathbb{R}|$:

1. $|\mathbb{R}| = |\text{any interval}|$ use function like $\tan(x)$

use linear functions to switch between different intervals.



$\tan(x)$:
$(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

2. Use (without proof is ok) that: $|\mathbb{R}| = |\{\text{sequence of } 0, 1\}|$

$= |\{\text{sequence of } 0, 1, \cdots 9\}|$

decimal expansions

(works for $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$)

3. $|\text{uncountable set} - \text{Countable set}| = |A|$

A

$|A - B| = |A|$

A: any uncountable set

B: countable set.

- Function:

  - bijective function between two open intervals, use linear
    $f: (a,b) \to (c,d)$
    $y = mx + k$
    find m, k.

  - from open interval to closed interval
    $(a,b) \to [c,d]$
    use (3)

  - #8 from Dec 2013
    $f: [-1,1] \setminus \{0\} \to \mathbb{R}$
    $f(x) = \frac{1}{x} - x$ prove it is a bijective and then find f^{-1}.

1) injective: want to prove if $x_1 \neq x_2$ then $\frac{1}{x_1} - x_1 \neq \frac{1}{x_2} - x_2$

Contrapositive: suppose $\frac{1}{x_1} - x_1 = \frac{1}{x_2} - x_2$
given that $x_1, x_2 \neq 0$

$$x_2(1 - x_1^2) = (1 - x_2^2)x_1$$
$$x_2 - x_1 + x_2 x_1 (x_2 - x_1) = 0$$
$$(x_2 - x_1)\underbrace{(1 + \underbrace{x_2 x_1}_{|x_1 x_2| < 1})}_{\neq 0} = 0$$
$$x_2 - x_1 = 0$$
$$x_1 = x_2$$

Try: maybe f is decreasing?
$f'(x) = -\frac{1}{x^2} - 1 < 0$

Also, could prove algebraically: $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$

Surjective? let $y \in \mathbb{R}$, Need to prove $\exists x \in (-1, 1] \setminus \{0\}$
s.t. $\frac{1}{x} - x = y$, solve for x.
show your x lies in $(-1, 1] - \{0\}$

> Alternative: $\lim\limits_{x \to 0^-} f(x) = -\infty$
>
> $\lim\limits_{x \to 0^+} f(x) = +\infty$
>
> $\lim\limits_{x \to -1} f(x) = 0$
>
> use I.V.T.

For $f^{-1}(y)$:
we wanted x s.t. $f(x) = y$

$$x = -\frac{y}{2} \pm \sqrt{\frac{y^2}{4} + 1}$$

Recall: Need $x \in (-1, 1] - \{0\}$
when $y \geq 0$, then $-\sqrt{\frac{y^2}{4}+1} < -1$
when $y < 0$ ....
should be only one should work.

- Congruence.
  - If $a \equiv 1 \mod 2$, $b \equiv 3 \mod 4$
    then $a^2 + b \equiv 0 \mod 4$ } #6 Dec 2013

Lemma: $a^2 \equiv 1 \mod 2$, then $a^2 \equiv 1 \mod 4$

2 ways: ① $a = 2k+1$ $k \in \mathbb{Z}$ b/c $a \equiv 1 \mod 2$

Then $a^2 = (2k+1)^2 = \underbrace{4k^2 + 4k}_{\equiv 0} + 1 \equiv 1 \mod 4$

② $a \equiv 1 \mod 2$

then: $a \equiv 1 \mod 4$
or
$a \equiv 3 \mod 4$
$\Rightarrow$ $a^2 \equiv 1$ or $a^2 \equiv 9 \equiv 1$

Now: $a^2 \equiv 1 \mod 4$
$+)\ b \equiv 3 \mod 4$

$a^2 + b \equiv 1 + 3 \mod 4 \equiv 0 \mod 4$

- Question (Dec. 2011)

  p – prime $p > 3$
  Then, $p^2 \equiv 1 \mod 6$
  $p^2 \equiv 1$

  or $25 \equiv 1 \mod 6$

* start with:

Since p is prime
$p \not\equiv 0, 2, 4, 3 \mod 6$

lemma : proof:
- $p \not\equiv 0 \mod 6$ obvious.
- since p is odd, it cannot be $\equiv 2 \mod 6$
  $6k+2$, $6k+4$ are even.
- if $p = 6k+3$, then p is divisible by 3.
  it can't be b/c prim)

- Graphs
  - Connected
    - Graph with degrees of vertices 1, 2, 3, 4, 4, 3 does not exist
    - 1, 2, 3, 4, 4, 3, 3
      # edges : $\frac{1}{2} \times (1+2+3+4+4+3+3)$
  - In any group of people shaking hands, there are two who shook the same number of hands.
    (Any graph has to have 2 vertice of the same degrees)
    Let a be the number of vertices,
    possible degrees : $0, 1, 2, \cdots a-1$
    Cannot both occur!