This midterm has 6 questions on 8 pages

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (Last, First):  
Student Number:  
Signature:  

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1. Short answer questions.

(a) Give the definition of a well-ordered subset of $\mathbb{R}$.

**Solution.** A set $A \subset \mathbb{R}$ is called well-ordered if every subset of $A$ has a minimal element.

(b) Write the negation of the following statement:

“For every positive $\epsilon$ there exists a positive $\delta$ such that if $|x| < \delta$ then $|f(x)| < \epsilon$.”

**Solution.** We start negating this sentence, and get: “There exists a positive $\epsilon$ such that for every positive $\delta$, $|x| < \delta$ does not imply that $|f(x)| < \epsilon$”. If we recall the rules for negating the implication, it is tempting to write: “There exists a positive $\epsilon$ such that for every positive $\delta$, $x < \delta$ and $|f(x)| \geq \epsilon$”. Note, however, that this doesn’t mean the same thing as the sentence just above (and actually, unlike the above sentence, it is not a statement at all, but is an open sentence with $x$ being a variable). This happens because when we say “$|x| < \delta$ implies $|f(x)| < \epsilon$”, we mean that this happens for all $x$ that satisfy $|x| < \delta$. Thus, when we say “$|x| < \delta$ does not imply that $|f(x)| < \epsilon$”, we mean that there exists $x$ such that $|x| < \delta$ and $|f(x)| \geq \epsilon$. Putting it all together we get:

“There exists a positive $\epsilon$ such that for every positive $\delta$, there exists $x$ such that $|x| < \delta$ and $|f(x)| \geq \epsilon$.”

(c) Is the number $0.7(\pi - 3.1415)$ rational or irrational? (Include a short proof; you can assume without proof that $\pi$ is irrational).

**Solution.** It is irrational. Two ways to prove this: could refer to the results from class: the sum/difference of a rational and irrational number is irrational; the product of a non-zero rational number and an irrational number is irrational.

Alternatively, you could just use proof by contradiction (this is how we obtained those results, anyway): Let $a = 0.7(\pi - 3.1415)$. Suppose this number was rational. Note that the numbers $0.7$ and $3.1415$ are both rational. Then we have: $\pi = a \frac{10}{7} + 3.1415$ would have been rational as well – a contradiction (we are given that $\pi$ is irrational).
2. For a natural number \( k \), define the symbol \( \binom{k}{2} \) to mean the number \( \binom{k}{2} = \frac{k(k-1)}{2} \), and \( \binom{k}{3} \) to mean \( \binom{k}{3} = \frac{k(k-1)(k-2)}{6} \). Prove that for every natural number \( n \geq 2 \) we have:

\[
\sum_{i=2}^{n} \binom{i}{2} = \binom{n+1}{3}.
\]

**Solution.** Use mathematical induction.

Base case: \( n = 2 \). Then the sum on left-hand side has only one term: \( \binom{2}{2} \); the right-hand side equals \( \binom{3}{3} \). Both numbers equal 1 by definition, so the equality holds in this case.

Induction step: Suppose we know this statement is true for \( n = k \); want to prove it for \( n = k + 1 \). That is, the induction assumption is that

\[
\sum_{i=2}^{k} \binom{i}{2} = \binom{k+1}{3},
\]

and assuming this, we need to prove that

\[
\sum_{i=2}^{k+1} \binom{i}{2} = \binom{(k + 1) + 1}{3}.
\]

The left-hand side of the equality we want to prove can be written as: \( \sum_{i=2}^{k} \binom{i}{2} + \binom{k+1}{2} \), and then by induction assumption, it equals \( \binom{k+1}{3} + \binom{k+1}{2} \). Thus, it remains to prove that

\[
\binom{k+1}{3} + \binom{k+1}{2} = \binom{k+2}{3}.
\]

We do this by plugging in the definition. On the LHS, we have:

\[
\binom{k+1}{3} + \binom{k+1}{2} = \frac{(k+1)k(k-1)}{6} + \frac{(k+1)k}{2} = \frac{(k+1)(k(k-1) + 3k)}{6}.
\]

On the RHS, we have:

\[
\binom{k+2}{3} = \frac{(k+2)(k+1)k}{6}.
\]

Thus, we need to check that

\[
\frac{(k+1)(k(k-1) + 3k)}{6} = \frac{(k+2)(k+1)k}{6}.
\]

This is equivalent to: \( k(k - 1) + 3k = (k + 2)k \), which is clearly true. This proves the induction step, and thus finishes the proof of our statement, by the Principle of Mathematical Induction.
3. Prove that if \( n \) and \( m \) are odd integers, then \( n^2 - m^2 \equiv 0 \pmod{8} \).

**The intended solution.** The easiest way to solve this problem was by cases: consider the possible remainders \( \pmod{8} \). If \( n \) is odd, it can be congruent to 1, 3, 5 or 7 \( \pmod{8} \). Then \( n^2 \) is congruent \( \pmod{8} \) to, respectively, \( 1^2 = 1 \), or \( 3^2 = 9 \equiv 1 \pmod{8} \), or \( 5^2 = 25 \equiv 1 \pmod{8} \), or \( 7^2 = 49 \equiv 1 \pmod{8} \). Thus we see that in all the cases, if \( n \) is odd, then \( n^2 \equiv 1 \pmod{8} \). This also shows that \( m^2 \equiv 1 \pmod{8} \), since \( m \) is odd as well. Thus, \( n^2 - m^2 \equiv 1 - 1 = 0 \pmod{8} \).

**Alternative solution (more popular):** Since \( n \) and \( m \) are odd, we can write \( n = 2k + 1 \), \( m = 2\ell + 1 \) for some integers \( k, \ell \). By definition of congruence, we need to prove that \( n^2 - m^2 \) is divisible by 8. Then

\[
 n^2 - m^2 = (2k + 1)^2 - (2\ell + 1)^2 = 4(k^2 - \ell^2 + k - \ell).
\]

Note that one cannot stop here: this clearly shows that \( n^2 - m^2 \) is divisible by 4, but more work is needed to show that \( n^2 - m^2 \) is divisible by 8: namely, we need to prove that the integer \( k^2 - \ell^2 + k - \ell \) is even. So let us prove the lemma:

**Lemma.** For any two integers \( k, \ell \), the number \( k^2 - \ell^2 + k - \ell \) is even.

Proof of the Lemma: either by cases (you need to consider 3 cases: \( k, \ell \) both even, both odd, and one is even one is odd), or better, note that for any \( k \), \( k^2 - k \) is even (this can be proved by cases: \( k \) even/odd, or by considering congruence \( \pmod{2} \)). Then

\[
 k^2 - \ell^2 + k - \ell = (k^2 - k) + (\ell^2 - \ell) \quad \text{is even}.
\]

Now, by the Lemma, we have that there exists an integer \( q \) such that \( k^2 - \ell^2 + k - \ell = 2q \), and putting it all together, we get that

\[
 n^2 - m^2 = 4 \cdot 2q = 8q.
\]
4. (a) Prove that there exists a positive integer that is divisible by 3 and congruent to 1 mod 5. Prove that there also exists an integer divisible by 5 and congruent to 1 mod 3.

**Solution.** The number 6 is divisible by 3 and congruent to 1 mod 5; the number 10 is divisible by 5 and congruent to 1 mod 3. (This is proof of existence by demonstrating an example).

(b) Prove that every integer greater than 7 can be written as a sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.

*Hint: Use induction or strong induction; part (a) might be useful or not.*

**Solution 1.** Use strong induction. Note that if we know that some integer \( k \) can be written as \( k = 3a + 5b \) with \( a, b \geq 0 \), then it is easy to show that \( k + 3 \) can be written in this form: namely, \( k + 3 = 3(a + 1) + 5b \). With this observation, we set up the proof by strong induction.

Base cases: \( n = 8 = 3 + 5, n = 9 = 3 \cdot 3, n = 10 = 5 \cdot 2 \). Induction step: assume that for some \( n \geq 10 \), we have that for any integer \( k \) satisfying \( 7 < k \leq n \) there exist \( a, b \in \mathbb{N} \cup \{0\} \) such that \( k = 3a + 5b \). Then we need to prove that the number \( n + 1 \) can be represented in this form.

Proof: consider \( k = n - 2 \). Then since \( n \geq 10 \), we have \( 7 < k < n \), so the induction assumption holds and there exist nonnegative integers \( a, b \) such that \( k = 3a + 5b \). Then \( n + 1 = (n - 2) + 3 = 3a + 5b + 3 = 3(a + 1) + 5b \), and so we can take the integers \( a + 1 \) and \( b \), and they give the required representation of the number \( n + 1 \), completing the proof of the induction step. Then by the principle of strong induction, the statement holds for all \( n \).

**Solution 2.** Use the usual induction (this is where we get to use part (a)). Base case: \( n = 8 \).

Some preliminary notes on the induction step: Assume \( n \geq 8 \), and there exist integers \( a, b \geq 0 \) such that \( n = 3a + 5b \). We need to prove that there exist integers \( a', b' \geq 0 \) such that \( n + 1 = 3a' + 5b' \). Recall from Part (a) that we have: \( 1 = 3 \cdot 2 - 5 \), so

\[
n + 1 = n + 3 \cdot 2 - 5 \quad \text{by induction assumption} = 3a + 5b + 3 \cdot 2 - 5 = 3(a + 2) + 5(b - 1).
\]

Thus we could take \( a' = a + 2, b' = b - 1 \). The only issue is that we need to have \( b' \geq 0 \), so this argument works only if \( b \geq 1 \). Thus, we need to consider the case \( b = 0 \) separately. Here is the complete proof.

Proof of the induction step: Assume \( n \geq 8 \), and there exist integers \( a, b \geq 0 \) such that \( n = 3a + 5b \). We need to prove that there exist integers \( a', b' \geq 0 \) such that \( n + 1 = 3a' + 5b' \). By induction assumption, we have \( n = 3a + 5b \). Consider two cases: \( b = 0 \) or \( b > 0 \). If \( b = 0 \), then we have \( n + 1 = 3(a + 2) + 5(b - 1) \), let \( a' = a + 2 \), \( b' = b - 1 \). Since \( b > 0 \) and is an integer, we have \( b' \geq 0 \), and this completes the proof in this case.

If \( b = 0 \), then we have \( n = 3a \) for an integer \( a \), and since \( n \geq 8 \), we have \( a \geq 3 \). We can write (using the second statement of Part (a)): \( 1 = 5 \cdot 2 - 3 \cdot 3 \). Then

\[
n + 1 = n + 5 \cdot 2 - 3 \cdot 3 = 3a + 5 \cdot 2 - 3 \cdot 3 = 3(a - 2) + 5 \cdot 2.
\]
Then we take $a' = a - 3$, $b' = 2$. Since we had $a \geq 3$, the number $a'$ is non-negative, and the proof of the induction step is completed.

**Remark:** Note that this problem can be formulated in a funny and illuminating way: suppose after a banking crisis, a country has just two types of notes: $3$ notes and $5$ notes. Prove that any integral amount larger than $7$ can be paid.

Our two solutions lead to different algorithms of arriving at a way to pay out a given amount of $n$ dollars: in the first solution, you use at most two $5$ notes to make up a number between 8 and 10 that is congruent to $n \mod 3$, and then use $3$ notes for the rest (so if you need to pay a million dollars, most of it will be done by $3$ notes. Can you tell if one or two $5$ notes will be used?)

In the second solution, on the other hand, as we go from $n$ to $n + 1$, we either trade a five for two threes, or if we have no fives in the pay-out of $n$, we trade three $3$ notes for two fives. This is a much slower to implement algorithm. Can you tell how many 3s and 5s will you need to pay out a million dollars using this algorithm?

(Note that for a million, neither is the fastest way to the answer, since you clearly could just use all fives, but neither algorithm will lead to this combination).

Anyway, this illustrates how you can have two algorithms achieving the same end, one a lot faster than the other; and how understanding congruences might help find a faster one.
5. Prove that there do not exist integers \( a, b \) and \( c \) such that

\[
12345678910111213 = a^2 + 25b^2 + 5c^2.
\]

**Solution.** The key to this question is to consider congruence \( \mod 5 \). Since 5 and 25 are divisible by 5, the right-hand side is congruent to \( a^2 \mod 5 \). The left-hand side is congruent to its last digit \( \mod 5 \), which is 3.

Now we can do proof by contradiction: suppose such \( a, b, c \) exist. Then we would have that \( a^2 \equiv 3 \mod 5 \). The Lemma below shows that this is a contradiction.

**Lemma.** If \( a \) is an integer, then \( a^2 \) is congruent to 0, 1, or 4 \( \mod 5 \).

**Proof.** Proof of Lemma: By cases.

Case 1: \( a \equiv 0 \mod 5 \). Then \( a^2 \equiv 0^2 = 0 \mod 5 \).

Case 2: \( a \equiv 1 \mod 5 \). Then \( a^2 \equiv 1^2 = 1 \mod 5 \).

Case 3: \( a \equiv 2 \mod 5 \). Then \( a^2 \equiv 2^2 = 4 \mod 5 \).

Case 4: \( a \equiv 3 \mod 5 \). Then \( a^2 \equiv 3^2 \equiv 4 \mod 5 \).

Case 5: \( a \equiv 4 \mod 5 \). Then \( a^2 \equiv 4^2 \equiv 1 \mod 5 \).
6. A small country has a single airline called RemoteAir, and \( a \) airports in total; the airport in the capital has 21 regular RemoteAir flights to other airports, the Middleofnowhere airport has only one flight to another airport, and every other airport has 20 regular flights to other airports.

(a) If one airplane can serve two regular routes, how many airplanes does RemoteAir need to have in order to provide this service? (Your answer should be a function of \( a \)).

**Solution.** Think of this situation as a graph: vertices are airports and edges are the regular routes. Then the problem is saying that we need one airplane per two edges. So let us first find the number of edges (call it \( e \)). We know from class that \( 2e \) is the sum of the degrees of all vertices. We are given that there is one vertex of degree 1, one vertex of degree 21, and \( a - 2 \) vertices of degree 20. Then we have:

\[
2e = 1 + 21 + 20(a - 2) = 20a - 18.
\]

Then \( e = 10a - 9 \), and we need \( 5a - 4 \) airplanes (note that the number of edges is odd, so there will be one airplane left serving just one route).

(b) Prove that it is possible to get (by air) from the capital to Middleofnowhere, possibly with several connections, regardless of what \( a \) is.

**Solution.** In the language of graph theory, we need to prove that the Middleofnowhere and the capital belong to the same connected component of our graph. Prove it by contradiction. Suppose they belonged to different connected components. Then consider the component that has Middleofnowhere (recall – this is the sub-graph of our graph that consists of all vertices that you can possibly get to from Middleofnowhere.) It is, by itself, a graph; and it would have one vertex of degree 1 (the Middleofnowhere) and some vertices of degree 20. By our assumption, it cannot have the capital. Thus our connected component has only one vertex of odd degree (which is impossible in any graph – the number of vertices of odd degree has to be even), so we have arrived at a contradiction.