This midterm has 7 questions on 8 pages

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (Last, First): 

Student Number: 

Signature: 

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1. Short answer question: no proofs required.

(a) Write the converse, contrapositive and negation of the following statement:

For every integer \( n \), if \( n \) is divisible by 3 then \( n^2 \) is divisible by 3.

**Solution:**

- Converse: For every integer \( n \), if \( n^2 \) is divisible by 3, then \( n \) is divisible by 3.
- Contrapositive: For every integer \( n \), if \( n^2 \) is not divisible by 3 then \( n \) is not divisible by 3.
- Negation: There exists an integer \( n \) such that \( n \) is divisible by 3 and \( n^2 \) is not divisible by 3.

(b) Let \( A_n \) be the interval \([-n, 2 + \frac{4}{n^2})\) for \( n \in \mathbb{N} \). Find \( \bigcap_{n \in \mathbb{N}} A_n \) and \( \bigcup_{n \in \mathbb{N}} A_n \) (no proof is required).

**Solution:**

- \( \bigcap_{n \in \mathbb{N}} A_n = [-1, 2] \)
- \( \bigcup_{n \in \mathbb{N}} A_n = (-\infty, 6) \)

(c) Write \( \mathbb{R} - \mathbb{N} \) as a union of an indexed collection of sets where each set is an interval.

**Solution:** If we let \( I := \mathbb{N} \cup \{0\} \), let \( A_0 := (-\infty, 1) \) and, for every \( n \in \mathbb{N} \), let \( A_n := (n, n+1) \), then:

\[
\mathbb{R} - \mathbb{N} = \bigcup_{i \in I} A_i.
\]
2. Consider the following two statements:

1. \( \forall n \in \mathbb{N}, \exists z \in \mathbb{Z} \text{ such that } z = n \)
2. \( \exists z \in \mathbb{Z} \text{ such that } \forall n \in \mathbb{N}, z = n \)

One of the statements is true, and the other is false. Determine which is which and prove both of your answers.

**Solution:**

The first statement is true.

*Proof.* Since \( \mathbb{N} \subset \mathbb{Z} \), given any \( n \in \mathbb{N} \) we can simply choose \( z \) to be equal to \( n \).

The second statement is false.

*Proof.* We prove that its negation: ‘\( \forall z \in \mathbb{Z}, \exists n \in \mathbb{N} \text{ such that } z \neq n \)’ is true. Given any \( z \in \mathbb{Z} \) we can simply choose \( n := z^2 + 1 \). Since \( z^2 + 1 \in \mathbb{N} \) and since \( z \neq z^2 + 1 \) for any integer \( z \) (why not?) this completes the proof.
3. (a) Give the definition of the power set $\mathcal{P}(A)$ of a set $A$.

Solution:
See textbook.

(b) Let $A = \{1, 2, \{1, 2\}\}$. Determine whether the following statements are True or False (and provide a brief explanation why).

(a) $\{1, 2\} \subseteq A$.
(b) $\{1, 2\} \subseteq \mathcal{P}(A)$.
(c) $\{1, 2\} \in A$.
(d) $\{1, 2\} \in \mathcal{P}(A)$.

Solution:

(a) True because $1 \in A$ and $2 \in A$.
(b) False because $1 \notin \mathcal{P}(A)$ and $2 \notin \mathcal{P}(A)$.
(c) True because it is listed as an element of $A$.
(d) True because (a) was true.
4. Let \( A := \{ n \in \mathbb{N} : \exists z \in \mathbb{Z} \text{ such that } n = 2z + 1 \} \) and let \( B := \{ n \in \mathbb{N} : \exists k \in \mathbb{N} \text{ such that } n = 2k \} \). Determine the following:

1. \( A \cap B \),
2. \( A \cup B \),
3. \( A - B \), and
4. \( B - A \).

**Solution:** Notice that \( A \) coincides with the odd natural numbers while \( B \) coincides with the even natural numbers. As such:

1. \( A \cap B = \emptyset \),
2. \( A \cup B = \mathbb{N} \),
3. \( A - B = A \), and
4. \( B - A = B \).
5. Suppose that the following three statements are true:

   1. Rainbows are colourful.
   2. If it isn’t sparkly, then it must be extravagant.
   3. Colourful things are never extravagant.

What can you conclude about rainbows?

**Solution:**

Combining statements 1 and 3 we see that since rainbows are colourful and colourful things are never extravagant we must have that rainbows are never extravagant. Reinterpreting statement 2 by its contrapositive we learn that if something is not extravagant then it must be sparkly. Therefore, rainbows are also sparkly!
6. Let \( n \in \mathbb{Z} \). Prove that \( n^3 - 5n^2 + 13 \) is odd.

**Solutions:**

**Proof.** Let \( n \in \mathbb{Z} \) and consider the following two (exhaustive) cases:

- If \( n \) is even, then there is some \( k \in \mathbb{Z} \) such that \( n = 2k \). In this case,
  \[
  n^3 - 5n^2 + 13 = (2k)^3 - 5(2k) + 13 = 8k^3 - 10k + 12 + 1 = 2(4k^3 - 5k + 6) + 1
  \]
  and since \( 4k^3 - 5k + 6 \in \mathbb{Z} \) we conclude that \( n^3 - 5n^2 + 13 \) is odd.

- On the other hand, if \( n \) is odd, then there is some \( k \in \mathbb{Z} \) such that \( n = 2k + 1 \). In this case,
  \[
  n^3 - 5n^2 + 13 = (2k + 1)^3 - 5(2k + 1) + 13 = 8k^3 + 12k^2 + 6k + 1 - 10k - 5 + 13
  = 8k^3 + 12k^2 - 4k + 8 + 1 = 2(4k^3 + 6k^2 - 2k + 4) + 1
  \]
  and since \( 4k^3 + 6k^2 - 2k + 4 \in \mathbb{Z} \) we conclude that \( n^3 - 5n^2 + 13 \) is odd.
7. The symmetric difference of two sets $A$ and $B$ is defined to be:

$$A \triangle B = (A - B) \cup (B - A).$$

(a) Draw a Venn diagram to illustrate $A \triangle B$.
(b) Draw a Venn diagram for $(A \triangle B) \triangle C$.
(c) Prove that $\overline{A \triangle B} = (A \cap B) \cup (\overline{A} \cap \overline{B})$.
(d) Is it true that $\overline{A \triangle B} = \overline{A} \triangle \overline{B}$? Either prove or give a counterexample.

Solution:

(a) See figure at https://en.wikipedia.org/wiki/Symmetric_difference

(b) See figure at https://en.wikipedia.org/wiki/Symmetric_difference

(c) **Proof.** We will show that

$$\overline{A \triangle B} \subseteq (A \cap B) \cup (\overline{A} \cap \overline{B}) \text{ and } \overline{A \triangle B} \supseteq (A \cap B) \cup (\overline{A} \cap \overline{B}).$$

We start with the first inclusion. Let $x \in \overline{A \triangle B}$, i.e., suppose that $x \notin A \triangle B$. This means that $x \notin A - B$ and $x \notin B - A$, i.e., that ‘$x \notin A$ or $x \in B$’ and ‘$x \notin B$ or $x \in A$’. Let us now consider two cases:

- If $x \in A$, then, by the first alternative above we must also have that $x \in B$. Therefore, $x \in A \cap B$ and consequently $x \in (A \cap B) \cup (\overline{A} \cap \overline{B})$.
- If $x \notin A$, then, by the second alternative above we must also have that $x \notin B$. Therefore $x \in \overline{A} \cap \overline{B}$ and consequently $x \in (A \cap B) \cup (\overline{A} \cap \overline{B})$.

This completes the proof that $\overline{A \triangle B} \subseteq (A \cap B) \cup (\overline{A} \cap \overline{B})$.

Proceeding with the second inclusion, let $x \in (A \cap B) \cup (\overline{A} \cap \overline{B})$ i.e., suppose that $x \in A \cap B$ or that $x \in \overline{A} \cap \overline{B}$. We consider these two cases separately:

- If $x \in A \cap B$, then $x \in A$ and $x \in B$. In particular, $x \notin A - B$ (since $x \in B$) and $x \notin B - A$ (since $x \in A$). Therefore, $x \notin (A - B) \cup (B - A) = A \triangle B$ and consequently $x \in \overline{A \triangle B}$.
- If $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{A}$ and $x \in \overline{B}$, i.e., $x \notin A$ and $x \notin B$. In particular, $x \notin A - B$ (since $x \notin A$) and $x \notin B - A$ (since $x \notin B$). Therefore, $x \notin (A - B) \cup (B - A) = A \triangle B$ and consequently $x \in \overline{A \triangle B}$.

This completes the proof that $\overline{A \triangle B} \supseteq (A \cap B) \cup (\overline{A} \cap \overline{B})$. \qed

(d) It is false!

Almost any collection of sets gives a counterexample. For example, take the universal set $U = \{1, 2, 3\}$, and let $A = \{1, 2\}$, $B = \{2, 3\}$. Then $A \triangle B = \{1, 3\}$, $\overline{A} \triangle \overline{B} = \{2\}$, $\overline{A} = \{3\}$, $\overline{B} = \{1\}$, $\overline{A \triangle B} = \{1, 3\}$, and we see that $\overline{A \triangle B} \neq \overline{A} \triangle \overline{B}$. There were many other correct examples.

**Remark:** We can show that in general, when $A \cap B \neq \emptyset$, there exists an element of $\overline{A \triangle B}$ which is not an element of $\overline{A} \triangle \overline{B}$ to conclude that $\overline{A \triangle B}$ is not a subset of $\overline{A} \triangle \overline{B}$.

**Proof.** Recall from our proof in part (c) that $\overline{A \triangle B} \supseteq A \cap B$. In particular, there is an element $y \in \overline{A \triangle B}$ such that $y \in A$ and $y \notin B$. For this element, we also have that $y \notin \overline{A}$ and $y \notin \overline{B}$. In turn, this implies that $y \notin \overline{A - B}$ (because $y \notin \overline{A}$) and that $y \notin \overline{B - A}$ (because $y \notin \overline{B}$). Finally, we must conclude that $y \notin (\overline{A - B}) \cup (B - A) = \overline{A \triangle B}$. This completes the proof. \qed