



**3 marks**

1. This question is multiple choice: for each of the statements below, write ‘True’, ‘False’, or ‘Incorrectly written’. You do not have to give explanations.

(a)  $\emptyset \cup \mathbb{N} = \mathbb{N}$ .

**Solution:** This is true – the union of the empty set and any set  $A$  is the set  $A$ ; in particular, this is true for  $A = \mathbb{N}$ .

(b)  $1 \notin \{\{1\}, 2, 3\}$ .

**Solution:** True: the elements of the set on the right are: the numbers 2 and 3, and the set  $\{1\}$ ; the number 1 is not among them.

(c)  $3 \subseteq \mathbb{N}$ .

**Solution:** This is “incorrectly written” the notation  $\subseteq$  applies only when you have sets on both sides, and 3 cannot be a name of a set. So, this is not even a true/false statement, because it simply does not make sense.

(d)  $\infty \in \mathbb{R}$ .

**Solution:** Here both answers “false” and “incorrectly written” were accepted; the main thing is, the symbol “ $\infty$ ” can be used only as part of interval notation (or also as part of limit notation), but it is not a number!

(e)  $[1, 2] \cap [2, \infty) = \{2\}$ .

**Solution:** This is true.

**Solution:** A remark on marking: 1 point was taken off for every wrong answer. So if you got 2 out of 5 right, the score for this question is still 0.

**4 marks**

2. Construct the converse, contrapositive, and the negation of the statement

“If it is 5 o’clock, Her Majesty is having tea.”

**Solution:** Remark on marking: converse and contrapositive were 1 point each, negation was 2 points.

**Converse:** If H.M. is having tea, it is 5 o’clock.

**Contrapositive:** If H.M. is not having tea, it is not 5 o’clock.

**Negation:** It is 5 o’clock, and H.M. is not having tea.

**Most popular negation mistake:** “If it is 5 o’clock, H.M. is not having tea”. I feel compelled to include a discussion of why this is really wrong (apart from referring to the rules we talked about in class): “P does not imply Q” is not at all the same as “P implies not Q” – consider, for example, the statement “If I do well on the exam, it will snow in Vancouver tomorrow”. This statement is clearly false. So its negation must be true. But if you negate it in this fashion, you will get “If I do well on the exam, it will not snow in Vancouver tomorrow”, which is nowhere closer to true than the original statement. If you made any other mistake, also try the same with the sentence about snow in Vancouver, and see why it does not make sense.

4 marks

3. Using any method you like, prove that the following statement is a tautology (that is, it is true for any truth values of the statements  $P$ ,  $Q$ , and  $R$ ):

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R).$$

**Solution:** The most popular (accepted) solution was by truth tables. I will not reproduce it here. Note that the truth table had to have 8 rows, one for every possible combination of T/F values for  $P$ ,  $Q$ , and  $R$ .

My preferred solution:

In fact, the statements  $(P \Rightarrow (Q \Rightarrow R))$  and  $(P \wedge Q \Rightarrow R)$  are logically equivalent, and in particular, it is a tautology that one implies the other. Let us prove their logical equivalence.

$$\begin{aligned} P \Rightarrow (Q \Rightarrow R) &\equiv \sim P \vee (Q \Rightarrow R) \equiv \sim P \vee (\sim Q \vee R) \\ &\equiv (\sim P \vee \sim Q) \vee R \text{ (by associativity law)} \\ &\equiv \sim(P \wedge Q) \vee R \text{ (by DeMorgan's law)} \\ &\equiv (P \wedge Q) \Rightarrow R. \end{aligned}$$

□

**One more good solution (possibly my favourite):** we have to show that the implication  $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$  is true regardless of the truth values of  $P$ ,  $Q$ , and  $R$ . How can this implication be false? By definition of the implication, it would be false if  $P \Rightarrow (Q \Rightarrow R)$  was true and  $(P \wedge Q) \Rightarrow R$  was false. Now,  $(P \wedge Q) \Rightarrow R$  is false only when  $P \wedge Q$  is true, but  $R$  is false. But then  $P$  is true,  $Q$  is true, and  $R$  is false. Let us now look at the left-hand side. If  $P$  and  $Q$  are true, and  $R$  is false, then the implication  $Q \Rightarrow R$  is false (by definition of the implication); and then  $P \Rightarrow (Q \Rightarrow R)$  is also false. Thus, we have shown that it is impossible for the left-hand side to be true, but the right-hand side to be false, and so the implication is a tautology.

4 marks

4. Let  $A = \{x \in \mathbb{R} \mid \exists n \in \mathbb{Z} \text{ s.t. } n < x < n + 1\}$ .

(a) Write the set  $A$  as a union or intersection of an indexed collection of sets.

**Solution:** Let  $A_n = (n, n + 1)$ . Then

$$A = \bigcup_{n \in \mathbb{Z}} A_n = \bigcup_{n \in \mathbb{Z}} (n, n + 1).$$

(b) Find the complement of  $A$ . Explain.

**Solution:** The complement of  $A$  is the set of integers  $\mathbb{Z}$ .

One could see this by drawing the union of the intervals on the real line, and seeing that all points except for their common ends (that is, the integers) are covered. (At this point you should draw the line, and draw the intervals, if you have not yet done it).

All reasonable informal explanations were accepted, but here's the formal proof: We need to prove two inclusions:  $\overline{A} \subseteq \mathbb{Z}$  and  $\overline{A} \supseteq \mathbb{Z}$ .

The first inclusion: let  $x \notin A$ . Then, by definition,  $\nexists n \in \mathbb{Z}$  such that  $n < x < n + 1$ . However, note that for **every** real number  $x$  there exists an integer  $n$  such that  $n \leq x < n + 1$ . Then the only way that an integer  $n$  such that  $n < x < n + 1$  would not exist is if  $x = n$  for some integer  $n$ . The reverse inclusion: Let  $a \in \mathbb{Z}$  be an integer. Then the statement  $\exists n : n < a < n + 1$  is false, because for every  $n$ , there are no integers strictly between  $n$  and  $n + 1$ .

Note that doing the formal procedure of writing the complement of the union as intersection of complements is not very useful here: it was hard to get it right, and it gives a very confusing answer even if done right.

**4 marks**

5. (a) For the following statement, write the negation both in symbols and in words:

$$\exists(x, y) \in \mathbb{R} \times \mathbb{R}, \text{ s.t. } x^2 + y^2 = 1 \text{ and } x = y.$$

**Solution:** In symbols:  $\forall(x, y) \in \mathbb{R} \times \mathbb{R}, x^2 + y^2 \neq 1 \text{ or } x \neq y.$

In words: every point on the plane either does not lie on the unit circle  $x^2 + y^2 = 1$ , or does not lie on the line  $x = y$ .

**Most popular mistake:** using “such that” (or “s.t.”) in the negation, in combination with the quantifier  $\forall$ . If you made this mistake, please read the handouts on quantifiers that are posted on the website.

- (b) Sketch the set of points  $(x, y)$  that satisfy the condition from part (a) (if this set is not empty).

**Solution:** This set is the intersection of the unit circle with the line  $x = y$ . (it consists of two points:  $(\sqrt{2}/2, \sqrt{2}/2)$  and  $(-\sqrt{2}/2, -\sqrt{2}/2)$ ).

**6 marks**

6. Prove or disprove:

(a)  $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } xy \geq 0.$

**Solution:** This statement is true. There are two reasonable proofs.**Proof 1.** Let  $x$  be any real number. We need to show that there exists a real number  $y$  such that the product  $xy$  is non-negative. There are 3 cases:  $x > 0$ ,  $x < 0$ , and  $x = 0$ . In the case  $x > 0$ , take  $y = 1$ . Then  $xy = x > 0$ . In the case  $x < 0$ , take  $y = -1$ . Then  $xy = -x > 0$ . In the case  $x = 0$ , take  $y = 0$  (any number  $y$  would work), then  $xy = 0 \geq 0$ .**Proof 2.** Let  $x$  be any real number. Take  $y = 0$ . Then  $xy = 0 \geq 0$ . (So we showed that the number  $y = 0$  “works” for every real number  $x$ , and so the statement is true).**Very important remark:** Even though here it turned out that the same number  $y$  “works” for all  $x$ , one should not confuse the two statements:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } xy \geq 0. \text{ (that's the statement we were asked about),}$$

and

$$\exists y \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{R}, xy \geq 0.$$

Even though here both statements turn out to be true, these are very different statements. If not sure about this, please read the handouts on quantifiers on the website.

(b)  $\exists y \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{R}, xy < 0.$

**Solution:** Note that this statement is exactly the negation of the statement in part (a). Since the statement in (a) is true, this statement is false.

(c)  $\forall x, y \in \mathbb{R}, (x \geq y) \Rightarrow (x^2 \geq y^2).$

**Solution:** This statement is false, and to show that it is false, we just need to provide a counterexample – that is, find a pair of real numbers  $x$  and  $y$  such that  $x \geq y$  and  $x^2 < y^2$ . Take  $x = 0$ ,  $y = 1$ , for example.

**6 marks**

7. (a) Prove that if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

**Solution:** Let  $A$ ,  $B$ , and  $C$  be sets such that  $A \subseteq B$  and  $B \subseteq C$ . We need to show that then  $A \subseteq C$ . By definition of “subset”, that means, we need to show that every element of  $A$  is also an element of  $C$ . Let  $x \in A$ . Then since  $A \subseteq B$ , we have  $x \in B$ . Since  $B \subseteq C$ , this implies  $x \in C$ .  $\square$

**Remark:** Venn diagrams were useful as an illustration here, but they do not constitute a proof. 1 point was given for a Venn diagram without good proof.

- (b) Prove that if  $B \subseteq A$ , then  $B \cup A = A$ .

**Solution:** By definition  $B \cup A \supseteq A$  always, so we just need to show that if  $B \subseteq A$ , then  $B \cup A \subseteq A$ . Let  $x \in B \cup A$ . Then  $x \in A$  or  $x \in B$ , by definition of the union. If  $x \in A$ , then there is nothing to prove. If  $x \in B$ , then  $x$  is also in  $A$ , since  $B$  is a subset of  $A$ . Thus, in any case we have shown that if  $x \in B \cup A$ , then  $x \in A$ , so  $B \cup A \subseteq A$ .

Note: proof by contrapositive also worked here.

**4 marks**

8. Prove or disprove the following statement

Let  $n \in \mathbb{Z}$ . Then  $n$  is even if and only if  $5n^2 - 2n + 3$  is odd.

**Solution:** The implication “ $\Rightarrow$ ”. Let  $n$  be even. Then there exists  $k \in \mathbb{Z}$  such that  $n = 2k$ . Then

$$5n^2 - 2n + 3 = 5 \cdot 4k^2 - 2 \cdot 2k + 3 = 2(10k^2 - 2k + 1) + 1,$$

so  $5n^2 - 2n + 3$  is odd.

The implication “ $\Leftarrow$ ”. Proof by contrapositive. Let  $n$  be odd. Then  $n = 2k + 1$  for some integer  $k \in \mathbb{Z}$ . Then

$$5n^2 - 2n + 3 = 5 \cdot (4k^2 + 4k + 1) - 2 \cdot (2k + 1) + 3 = 20k^2 + 20k + 5 - 4k - 2 + 3 = 2(10k^2 + 8k + 4),$$

so  $5n^2 - 2n + 3$  is even.