This test has 3 questions on 4 pages, for a total of 28 points.

Duration: 45 minutes

- Write your name on every page.
- You need to show enough work to justify your answers.
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. A one-sided cheat sheet is allowed. Electronic devices of any kind (including calculators, cell phones, etc.) are NOT allowed.

Full Name (including all middle names): __________________________________________

Student-No: ________________________________________________________________

Signature: _________________________________________________________________

Section number/time of class: 11-12, section 202. __________________________________

Name of the instructor: ________________________________________________________

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1. Find the maximum volume of a box that fits into the part of the paraboloid \( z = 4 - x^2 - 9y^2 \) that lies above the \( xy \)-plane.

Solution: A maximal box that fits into the paraboloid would have to have a vertex on it (otherwise it could be enlarged); let such a vertex in the first octant have the coordinates \((x, y, z)\). Then the volume of the box is \((2x)(2y)z\). Thus, we are trying to maximize the function \( f(x, y, z) = 4xyz \) subject to the constraint \( g(x, y, z) = 4 - x^2 - 9y^2 - z = 0 \). Use Lagrange multipliers (we can just maximize the function \( xyz \) instead of \( 4xyz \) for simplicity of the calculation). We get:

\[
\begin{align*}
yz + 2\lambda x &= 0 \\
xz + 18\lambda y &= 0 \\
xy + \lambda &= 0 \\
4 - x^2 - 9y^2 - z &= 0.
\end{align*}
\]

If we multiply the first equation by \( x \) and the second equation by \( y \) and subtract, we get: \( 2\lambda x^2 - 18\lambda y^2 = 0 \), which gives \( x = \pm 3y \) or \( \lambda = 0 \). If \( \lambda = 0 \), we get \( xy = 0 \) from the third equation, which means the volume of the box would be zero; this clearly cannot be maximum (it actually would give minimum volume), so since we are looking for a maximum, we ignore this point and get \( x = 3y \) (We recall that our convention, when we made an association between a point on the paraboloid and a box, was \( x, y, z > 0 \), so we take \( x = 3y, y > 0 \), and proceed. Plug the third equation into the first equation: \( \lambda = -xy, yz - 2x^2y = 0 \), which gives \( z = 2x^2 \) or \( y = 0 \). Again, the solution \( y = 0 \) would give us a minimum point, not a maximum, so we discard it. Now plug \( z = 2x^2 = 18y^2 \) into the last equation. We get: \( 4 - 18y^2 - 18y^2 = 0 \), so \( 36y^2 = 4 \), \( y = 1/3 \) (since we are only interested in \( y > 0 \)). Then we get: \( x = 1, z = 2 \). We have found that the point with \( x, y, z > 0 \) on the paraboloid that gives the maximum \( xyz \) is \((1, 1/3, 2)\). Now, in principle we need to check also the points on the paraboloid where \( x \) or \( y \) or \( z = 0 \) (the boundary of the part of the paraboloid we just investigated), but as already noted, they would give minimal volume not maximal volume of a box. So the answer is:

\[
V_{\text{max}} = 4 \cdot 1 \cdot \frac{1}{3} \cdot 2 = \frac{8}{3}.
\]
2. Let $D$ be the (bounded) domain in $\mathbb{R}^2$ bounded by the parabolas $x = y^2$ and $x = 9 - 2y^2$.

(a) Sketch $D$.

**Solution:** We need to find the intersection points of the two parabolas: $y^2 = 9 - 2y^2 \iff 3y^2 = 9 \iff y = \pm \sqrt{3}$. (The corresponding value of $x$ is 3.)

![Diagram of the domain](image)

(b) Fill in the limits:

$$\iint_D ye^x \, dA = \int \int ye^x \, dxdy$$

and evaluate the integral.

**Solution:**

$$\iint_D ye^x \, dA = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{y^2}^{9-2y^2} ye^x \, dxdy.$$  

(Note that the limits for $y$ were found above in Part (a).) We evaluate:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{y^2}^{9-2y^2} ye^x \, dxdy = \int_{-\sqrt{3}}^{\sqrt{3}} y e^{x=9-2y^2} \, dy = \int_{-\sqrt{3}}^{\sqrt{3}} y (e^{9-2y^2} - e^{y^2}) \, dy = 0.$$  

(There are two ways to see that the last integral is zero: you could note that we are integrating an odd function of $y$ over a symmetric interval, or do a substitution $u = y^2$ (we have, conveniently, $ydy = \frac{1}{2}du$), and arrive at the same answer.)

(c) Write the same integral in the order

$$\iint ye^x \, dydx$$

(do not evaluate again).

**Solution:** We see from Part (a) that the integral will split at $x = 3$. More precisely, when $0 \leq x \leq 3$, we are integrating along $y$ from the bottom branch of the parabola $x = y^2$ to the top branch; when $3 \leq x \leq 9$, we are integrating from the bottom branch of the parabola $x = 9 - 2y^2$ to the top branch. We need to rewrite the equations of these parabolas so that $y$ is expressed in terms of $x$. The first one is: $y = \pm \sqrt{x}$, and the second one is: $y = \pm \sqrt{(9-x)/2}$. We get the answer:

$$\int_0^3 \int_{-\sqrt{x}}^{\sqrt{x}} ye^x \, dydx + \int_3^9 \int_{-\sqrt{(9-x)/2}}^{\sqrt{(9-x)/2}} ye^x \, dydx.$$
3. Find the $y$-coordinate of the centroid of the lamina that is shaped as the part of the cardioid given by the equation $r = 2(1 - \cos(\theta))$ that lies in the second quadrant (to the left of the $y$-axis and above the $x$-axis).

**Solution:** Since we are interested in the part that lies in the second quadrant, we have $\pi/2 \leq \theta \leq \pi$. Then the total mass of the lamina (with density 1 since we are looking for the centroid, so mass is the same as total area) is:

$$M = \int_{\pi/2}^{\pi} \int_{0}^{2(1 - \cos(\theta))} 1 \cdot r \, dr \, d\theta;$$

and (using that $y = r \sin \theta$),

$$\bar{y} = \frac{1}{M} \int_{\pi/2}^{\pi} \int_{0}^{2(1 - \cos(\theta))} r \sin \theta \, dr \, d\theta.$$

First evaluate $M$:

$$M = \int_{\pi/2}^{\pi} \int_{0}^{2(1 - \cos(\theta))} r \, dr \, d\theta = \frac{1}{2} \int_{\pi/2}^{\pi} r^2 \left( 1 - \cos(\theta) \right) d\theta$$

$$= 2 \int_{\pi/2}^{\pi} (1 - \cos \theta)^2 d\theta = 2 \int_{\pi/2}^{\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta$$

$$= 2 \left( \frac{\pi}{2} - 2 \sin \theta \big|_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \frac{1 + \cos(2\theta)}{2} d\theta \right) = \pi + 4 + \frac{\pi}{2} + 0 = 4 + \frac{3\pi}{2}.$$

Now evaluate the integral for $\bar{y}$:

$$\int_{\pi/2}^{\pi} \int_{0}^{2(1 - \cos(\theta))} r^2 \sin \theta \, dr \, d\theta = \frac{1}{3} \int_{\pi/2}^{\pi} \sin \theta r^3 \left( 1 - \cos(\theta) \right) d\theta$$

$$= \frac{8}{3} \int_{1}^{2} u^3 \, du = \frac{2}{3} (16 - 1) = 10,$$

where the substitution was: $u = 1 - \cos(\theta)$, $du = \sin \theta \, d\theta$.

Final answer:

$$\bar{y} = \frac{10}{4 + \frac{3\pi}{2}}.$$