This midterm has 4 questions on 7 pages, for a total of 30 points.

Duration: 40 minutes

- Write your name on every page.
- You need to show enough work to justify your answers.
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. Electronic devices of any kind (including calculators, cell phones, etc.) are NOT allowed.

Full Name (including all middle names): ____________________________

Student-No: ________________________________________________

Signature: ________________________________________________

Section number: ____________________________________________

Name of the instructor: _______________________________________

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1. (a) Find a vector of length 2 forming an angle $\pi/2$ with $\langle 3, 2 \rangle$.

we want: $\vec{v} \perp \langle 3, 2 \rangle$ and $|\vec{v}| = 2$.

First find a vector $\vec{w}$ orthogonal to $\langle 3, 2 \rangle$; for example, $\vec{w} = \langle -2, 3 \rangle$ satisfies $\vec{w} \cdot \langle 3, 2 \rangle = 0$.

Now make a vector of length 2 parallel to $\vec{w}$:

$$\vec{v} = \frac{-2}{\sqrt{13}} \cdot 2 = \langle -2, 3 \rangle \cdot \frac{2}{\sqrt{13}} = \left[ \frac{-4}{\sqrt{13}}, \frac{6}{\sqrt{13}} \right].$$

(b) Let $\vec{w} = \langle 3, 4, 5 \rangle$. Find the vector projection $\text{proj}_\vec{w} \langle 1, 2, 2 \rangle$.

$$\text{proj}_\vec{w} \langle 1, 2, 2 \rangle = \left( \frac{\langle 1, 2, 2 \rangle \cdot \vec{w}}{|\vec{w}|} \right) \frac{\vec{w}}{|\vec{w}|}$$

$$= \langle 1, 2, 2 \rangle \cdot \langle 3, 4, 5 \rangle \cdot \frac{1}{3^2 + 4^2 + 5^2} \cdot \langle 3, 4, 5 \rangle = \frac{21}{50} \langle 3, 4, 5 \rangle$$

(c) Let $\vec{w} = \langle 3, 4, 5 \rangle$ as above. Find two vectors $\vec{v}_1$ and $\vec{v}_2$ such that $\vec{v}_1 - \vec{v}_2 = \langle 1, 2, 2 \rangle$, and $\vec{v}_1$ is parallel to $\vec{w}$, and $\vec{v}_2$ is perpendicular to $\vec{w}$.

we know that $\vec{v} - \text{proj}_w \vec{v}$ is perpendicular to $\vec{w}$.

so we can take $\vec{v}_1 = \text{proj}_w \vec{v} = \frac{21}{50} \langle 3, 4, 5 \rangle$ - from (b)

and $\vec{v}_2 = - (\vec{v} - \text{proj}_w \vec{v})$

Then $\vec{v}_1 - \vec{v}_2 = \vec{v}_1 + (\vec{v} - \text{proj}_w \vec{v}) = \vec{v}$

Answer: $\vec{v}_1 = \frac{21}{50} \langle 3, 4, 5 \rangle$ - from (b)

$\vec{v}_2 = \frac{21}{50} \langle 3, 4, 5 \rangle - \langle 1, 2, 2 \rangle = \langle \frac{13}{50}, -\frac{16}{50}, \frac{5}{50} \rangle$

(d) Let $\vec{v}_1, \vec{v}_2$ be any two vectors in $\mathbb{R}^3$ that lie in the $xy$-plane, and let $\vec{v}_3$ be any vector that does not lie in this plane. Find a plane that contains $(\vec{v}_1 \times \vec{v}_2) \times \vec{v}_3$.

$\vec{h} = \vec{v}_1 \times \vec{v}_2$ is perpendicular to the plane containing $\vec{v}_1, \vec{v}_2$.

$\vec{h} \times \vec{v}_3$ is perpendicular to $\vec{h}$ and $\vec{v}_3$.

since it is perpendicular to $\vec{h}$, it is contained in the plane containing $\vec{v}_1, \vec{v}_2$.

Answer: the plane spanned by $\vec{v}_1, \vec{v}_2$.
2. Consider the lines given in parametric form by:

\[ L_1 : \mathbf{r}_1(t) = (2, 5, 1) + t(3, 0, 4) \text{ and } L_2 : \mathbf{r}_2(t) = (-1, 2, 0) + t(6, 0, 8). \]

2 marks  
(a) Write \( L_2 \) in symmetric form.

\[ \begin{align*}
\frac{x + 1}{6} &= \frac{2}{8} \\
y &= 2
\end{align*} \]

3 marks  
(b) Find the distance from the point \((-1, 0, 2)\) to \( L_1 \).

We use the formula \( d = \frac{|\mathbf{PA} \times \mathbf{u}|}{|\mathbf{u}|} \), where \( \mathbf{v} = \) direction vector of the line, \( \mathbf{P} = (-1, 0, 2) \) - any point on the line.

Take \( \mathbf{A} = (2, 5, 1) \)

\[ \mathbf{PA} = (-3, -5, 1) \]
\[ \mathbf{v} = (3, 0, 4) \]
\[ \mathbf{PA} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -5 & 1 \\ 3 & 0 & 4 \end{vmatrix} = -20\mathbf{i} + 15\mathbf{j} - 15\mathbf{k} \]

\[ d = \frac{|-20 - 15 - 15|}{\sqrt{3^2 + 4^2}} = 5\sqrt{\frac{5^2 + 3^2 + 3^2}{5}} = \sqrt{34} \]

Different method of solving it:

\[ d = |\mathbf{PA} - \text{proj}_\mathbf{v} \mathbf{PA}| \]

\[ = |(-3, -5, 1) - (-\frac{3}{5}, 0, -\frac{4}{5})| \]

\[ = |\left(-\frac{12}{5}, -5, \frac{9}{5}\right)| \]

(agree with the previous answer)
(c) Do the lines $L_1$ and $L_2$ intersect?

Direction vector of $L_1$: $<3, 0, 4>$
Direction vector of $L_2$: $<6, 0, 8>$

These vectors are parallel, so $L_1$ and $L_2$ are parallel.

(d) Does there exist a plane containing both $L_1$ and $L_2$? If yes, find its equation.

Yes, there exists a plane containing these two parallel lines.

Let $P \in L_1$, $Q \in L_2$

The normal vector of our plane is perpendicular to the vector $\overrightarrow{PQ}$ and the (common) direction vector of $L_1$ and $L_2$.

We get:

$$\overrightarrow{n} = \overrightarrow{PQ} \times <3, 0, 4> = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -5 & 1 \\ 3 & 0 & 4 \end{vmatrix} = \begin{pmatrix} -20 \cr 15 \cr -15 \end{pmatrix}$$

Answer:

$$-20(x-2) + 15(y-5) - 15(z-1) = 0$$

or, cancelling $-5$,

$$4(x-2) - 3(y-5) + 3(z-1) = 0.$$
3. Describe the set of points \( P = (x, y, z) \) such that the distance from \( P \) to the plane \( x = y \) equals twice the distance from \( P \) to the plane with the equation \( x + z = 0 \).

The distance from \( P = (x, y, z) \) to \( x = y \) is

\[
d_1 = \frac{\langle x, y, z \rangle \cdot \vec{n}_1}{|\vec{n}_1|},
\]

where \( \vec{n}_1 \) is a normal vector of the plane \( x = y \).

(because this plane contains \((0,0,0)\))

Similarly, \( d_2 = \frac{\langle x, y, z \rangle \cdot \vec{n}_2}{|\vec{n}_2|} \), where \( \vec{n}_2 \) is the normal vector of \( x + z = 0 \).

We have:

\[
\vec{n}_1 = \langle 1, -1, 0 \rangle
\]

\[
\vec{n}_2 = \langle 1, 0, 1 \rangle
\]

(from the equations of the two planes.)

We want:

\[
d_1 = 2d_2
\]

This means:

\[
\frac{\langle x, y, z \rangle \cdot \langle 1, -1, 0 \rangle}{\sqrt{2}} = 2 \frac{\langle x, y, z \rangle \cdot \langle 1, 0, 1 \rangle}{\sqrt{2}}
\]

\[
x - y = 2(x + z)
\]

we get the plane \( x + y + 2z = 0 \)
4. (a) Let \( l \) be the line that lies in the \( yz \)-plane and is given by the equation \( 2z + 3y = 6 \) in the \( yz \)-coordinates. Find an equation of a plane containing the line \( l \) and parallel to the \( x \)-axis. Sketch \( l \) and sketch your plane.

\[ \frac{1}{2} x + \frac{3\sqrt{13}}{2} (y-2) + \sqrt{\frac{3}{13}} z = 0 \]  

[using the point \((0, 2, 0)\)]

The equation is the same:
\[ 2z + 3y = 6 \] in space, it defines a plane parallel to the \( x \)-axis.

(b) Find an equation of a plane containing the same line \( l \) and forming the angle \( \pi/3 \) with the \( yz \)-plane.

Let \( \vec{n} \) be a unit normal vector to this plane. Then we have:
\[ \vec{n} \cdot \vec{v} = 0 \] where \( \vec{v} \) is the direction vector of \( l \)

\[ \vec{n} \cdot \langle 1, 0, 0 \rangle = |\vec{n}| \cos \frac{\pi}{3} \]

\[ \vec{n} \] normal to \( yz \)-plane

because the angle with \( yz \)-plane (angle between normals) is \( \pi/3 \).

Answer:

If we take \( \vec{n} = \langle a, b, c \rangle \) to be a unit vector, we get the equations:

\[ \langle a, b, c \rangle \cdot \langle 0, 2, -3 \rangle = 0 \]
\[ \langle a, b, c \rangle \cdot \langle 1, 0, 0 \rangle = \frac{1}{2} \] from the angle
\[ a^2 + b^2 + c^2 = 1 \] \( \vec{n} \) is unit.

Thus:
\[ \begin{align*}
  a &= \frac{1}{2} \\
  2b - 3c &= 0 \\
  a^2 + b^2 + c^2 &= 1
\end{align*} \]

\[ \begin{align*}
  a &= \frac{1}{2} \\
  b &= \frac{3c}{2} \\
  c &= \pm \sqrt{\frac{3}{13}}
\end{align*} \]