This midterm has 4 questions on 7 pages, for a total of 30 points.

Duration: 40 minutes

- Write your name on every page.
- You need to show enough work to justify your answers.
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. Electronic devices of any kind (including calculators, cell phones, etc.) are NOT allowed.

Full Name (including all middle names): ________________________________

Student-No: ________________________________

Signature: ________________________________

Section number: ________________________________

Name of the instructor: ________________________________

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1. (a) Find a unit vector perpendicular to \( \langle 3, 2 \rangle \).

Let \( \vec{u} = \langle a, b \rangle \).

We have: \( \vec{u} \cdot \langle 3, 2 \rangle = 0 \), so \( 3a + 2b = 0 \).

We can take \( a = -2 \), \( b = 3 \): \( \langle -2, 3 \rangle \) perpendicular to \( \langle 3, 2 \rangle \).

Now, we make it into a unit vector:

\[
\vec{u} = \frac{\langle -2, 3 \rangle}{\sqrt{(-2)^2 + 3^2}} = \frac{\langle -2, 3 \rangle}{\sqrt{4 + 9}} = \frac{\langle -2, 3 \rangle}{\sqrt{13}} = \langle \frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \rangle
\]

(b) Let \( \vec{w} = \langle 1, 2, 2 \rangle \). Find the vector projection \( \text{proj}_w \langle 3, 4, 5 \rangle \).

Let \( \vec{u} = \frac{\vec{w}}{||\vec{w}||} = \frac{\langle 1, 2, 2 \rangle}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\langle 1, 2, 2 \rangle}{\sqrt{1 + 4 + 4}} = \frac{\langle 1, 2, 2 \rangle}{3} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle \).

\[
\text{Proj}_w \langle 3, 4, 5 \rangle = \frac{\langle 3, 4, 5 \rangle \cdot \vec{w}}{||\vec{w}||} \cdot \frac{\vec{w}}{||\vec{w}||} = \left( \langle 3, 4, 5 \rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle \right) \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle
\]

\[
= \left( 3 \cdot \frac{1}{3} + 4 \cdot \frac{2}{3} + 5 \cdot \frac{2}{3} \right) \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle = \left( 1 + \frac{8}{3} + \frac{10}{3} \right) \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle = \frac{21}{3} \cdot \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle = \langle \frac{7}{9}, \frac{4}{9}, \frac{4}{9} \rangle
\]

\[
= \langle \frac{7}{9}, \frac{1}{3}, \frac{1}{3} \rangle
\]

(c) Let \( \vec{w} = \langle 1, 2, 2 \rangle \) as above. Find two vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) such that \( \vec{v}_1 + \vec{v}_2 = \langle 3, 4, 5 \rangle \), and \( \vec{v}_1 \) is parallel to \( \vec{w} \), and \( \vec{v}_2 \) is perpendicular to \( \vec{w} \).

We know that \( \vec{v} - \text{proj}_w \vec{v} \) is perpendicular to \( \vec{w} \).

So, \( \vec{v}_1 = \text{proj}_w \vec{v} \) — from (b)

\[
\vec{v}_2 = \vec{v} - \text{proj}_w \vec{v}
\]

Answer:

\[
\vec{v}_1 = \langle \frac{7}{3}, \frac{14}{3}, \frac{14}{3} \rangle
\]

\[
\vec{v}_2 = \langle 3, 4, 5 \rangle - \langle \frac{7}{3}, \frac{14}{3}, \frac{14}{3} \rangle = \langle \frac{2}{3}, -\frac{2}{3}, -\frac{1}{3} \rangle
\]
(d) Find the volume of the parallelepiped defined by the vectors \((1, 2, 3), (1, 0, 1)\) and \((0, 3, 5)\).

\[
V = \left| \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 3 & 5 \end{vmatrix} \right| = \left| \langle 1, 2, 3 \rangle \times \langle 1, 0, 1 \rangle \cdot \langle 0, 3, 5 \rangle \right|
\]

\[
\langle 1, 2, 3 \rangle \times \langle 1, 0, 1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 2\hat{i} + 2\hat{j} - 2\hat{k}
\]

\[
\langle 1, 2, 3 \rangle \times \langle 1, 0, 1 \rangle \cdot \langle 0, 3, 5 \rangle = \langle 2, 2, -2 \rangle \cdot \langle 0, 3, 5 \rangle
\]

\[
= 6 - 10 = -4.
\]

\[
V = |-4| = 4.
\]
2. Consider the lines given in parametric form by:

\[ L_1 : \mathbf{r}_1(t) = (2, 5, 1) + t(1, 2, 0) \quad \text{and} \quad L_2 : \mathbf{r}_2(t) = (-1, 2, 0) + t(6, 6, 2). \]

2 marks  
(a) Write \( L_1 \) in symmetric form.

First, write it in coordinate form:

\[
\begin{align*}
&x = 2 + t \\
y = 5 + 2t \\
z = 1
\end{align*}
\]

symmetric:

\[
\begin{align*}
x - 2 &= \frac{y - 5}{2} \\
z &= 1
\end{align*}
\]

3 marks  
(b) Find the distance from the point \((-1, 0, 2)\) to \( L_1 \).

Two ways to do it:

1. \[ P (\mathbf{a}) = (-1, 0, 2) \]

\[ \mathbf{a} = (2, 5, 1) \]

\[ \mathbf{v} = (1, 2, 0) \]

- direction vector of \( L_1 \).

we can use the formula

\[ d = \left| \mathbf{PA} - \text{proj}_{\mathbf{v}} \mathbf{PA} \right| \]

\[ = \left| \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right| \]

\[ = \left| \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} + \frac{13}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right| \]

\[ = \left| \begin{pmatrix} -\frac{2}{5} \\ -\frac{1}{5} \\ 1 \end{pmatrix} \right| = \frac{\sqrt{4 + 1 + 25}}{5} = \frac{\sqrt{30}}{5} \]

(2) Different solution:

use the formula \[ d = \left| \frac{\mathbf{PA} \times \mathbf{v}}{15} \right| = \left| \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right| \]

\[ \hat{i} \hat{j} \hat{k} \\
-3 -5 1 \\
1 2 0 \\
= -2 \hat{i} + \hat{j} - \hat{k} \]

\[ = \frac{\sqrt{4 + 1 + 1}}{\sqrt{5}} = \frac{\sqrt{6}}{\sqrt{5}} \]

Note: \[ \frac{\sqrt{30}}{5} = \frac{\sqrt{6}}{\sqrt{5}} \]
(c) Find the intersection point of \( L_1 \) and \( L_2 \).

To find the intersection point, we need to rename the variable for \( L_2 \):

\( L_1 \) is given by \( \begin{cases} x = 2 + t \\ y = 5 + 2t \\ z = 1 \end{cases} \)  

\( L_2 \) is given by \( \begin{cases} x = -1 + 6s \\ y = 2 + 6s \\ z = 2s \end{cases} \)  

Intersection point:

\[
\begin{align*}
2 + t &= -1 + 6s \\
5 + 2t &= 2 + 6s \\
1 &= 2s
\end{align*}
\]

Then \( t = 0 \) gives us

the intersection point: \( (2, 5, 1) \)

(d) Does there exist a plane containing both \( L_1 \) and \( L_2 \)? If yes, find its equation.

we have two intersecting lines. They determine a plane.

normal \( \vec{n} \) to this plane is perpendicular to both direction vectors.

\[
\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 3 & 3 & 1 \end{vmatrix} = 2\vec{i} - \vec{j} - 3\vec{k}
\]

so we have:

we have the intersection point \( (2, 5, 1) \) from (c)

(or could use any point on either line)

Answer:

\[
2(x - 2) = (y - 5) - 3(z - 1) = 0
\]
3. Describe the set of points in space that are equidistant from the plane \( x = y \) and the plane with the equation \( x + z = 0 \).

The first plane has normal vector \( \vec{v}_1 = \langle 1, 1, 0 \rangle \)

The second plane has normal \( \vec{v}_2 = \langle 1, 0, 1 \rangle \)

The formula for the distance from a point \( (x_1, y_1, z_1) \) to the first plane gives:

\[
\langle x_1, y_1, z_1 \rangle \cdot \frac{\vec{v}_1}{\|\vec{v}_1\|} = d_1
\]

Vector from \((0,0,0)\) to point in the plane \( x = y \)

Vector from \((0,0,0)\) to our point \((x_1, y_1, z_1)\)

Similarly, the distance from \((x_1, y_1, z_1)\) to the second plane is

\[
d_2 = \langle x_1, y_1, z_1 \rangle \cdot \frac{\vec{v}_2}{\|\vec{v}_2\|}
\]

We get: \( d_1 = d_2 \) means:

\[
\left| \langle x_1, y_1, z_1 \rangle \cdot \langle 1, 1, 0 \rangle \right| = \left| \langle x_1, y_1, z_1 \rangle \cdot \langle 1, 0, 1 \rangle \right|
\]

\[
|x-y| = |x+z| \therefore x-y = \pm (x+z)
\]

and

\[
\begin{align*}
2 + y &= 0 \\
x + z - y &= 0
\end{align*}
\]

Answer: The planes

\[
2 = -y, \quad \text{and} \quad 2x + z - y = 0
\]
4. (a) Let \( \ell \) be the line that lies in the \( xy \)-plane and is given by the equation \( 2x + 3y = 6 \) in the \( xy \)-coordinates. Find an equation of a plane containing the line \( \ell \) and perpendicular to the \( xy \)-plane. Sketch \( \ell \) and sketch your plane.

Our plane is perpendicular to the \( xy \)-plane, so it is parallel to the \( z \)-axis. So \[2x + 3y = 6\] is the equation of the required plane.

(b) Find an equation of a plane containing the same line \( \ell \) and forming the angle \( \pi/6 \) with the \( xy \)-plane.

Let \( \overrightarrow{n} \) be the normal vector to our plane. Then we know: \( \overrightarrow{n} \) is perpendicular to \( \ell \) and \( \overrightarrow{n} \) forms the angle \( \pi/6 \) with \( \langle 0,0,1 \rangle \langle 0,0,1 \rangle \) normal of the \( xy \)-plane.

Let \( \overrightarrow{n} = \langle a, b, c \rangle \). Assume \( \overrightarrow{n} \) is unit.

We get: \[ \langle a, b, c \rangle \cdot \langle 0,0,1 \rangle = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \]

So: \[\langle a, b, c \rangle \cdot \langle -3, 2, 0 \rangle = 0\]

\[ c = \frac{\sqrt{3}}{2} \quad \text{from the angle with the \( xy \)-plane} \]

\[ a^2 + b^2 + c^2 = 1 \quad \text{\( \overrightarrow{n} \) is unit.} \]

\[ \begin{cases} -3a + 2b = 0 \\ c = \frac{\sqrt{3}}{2} \\ a^2 + b^2 + c^2 = 1 \end{cases} \]

Answer: \[ \langle \frac{1}{\sqrt{13}}, \frac{3}{2\sqrt{13}}, \frac{\sqrt{3}}{2} \rangle \]

\[ \overrightarrow{n} = \langle \frac{1}{\sqrt{13}}, \frac{3}{2\sqrt{13}}, \frac{\sqrt{3}}{2} \rangle \]

\[ \frac{13}{4}a^2 = \frac{1}{4} \quad \text{and} \quad a = \pm \frac{1}{\sqrt{13}} \]