Worksheet 1: skew lines in space

Let the lines $L_1$ and $L_2$ be given by the parametric equations

$$r_1(t) = ti + (1 - 2t)j + (2 + 3t)k,$$
$$r_2(s) = (3 - 4s)i + (2 + 3s)j + (1 - 2s)k.$$

**Question 1:** do these lines intersect?

**Solution:** These two lines intersect if there exist values of $t$ and $s$ such that $r_1(t) = r_2(s)$. Equating each of the three components, we get three equations, which we try to solve for $t$ and $s$. If a simultaneous solution exists, the lines intersect; if not, they don’t.

$$\begin{cases}
  t = 3 - 4s \\
  1 - 2t = 2 + 3s \\
  2 + 3t = 1 - 2s.
\end{cases}$$

Plugging in $t = 3 - 4s$ from the first equation into the second one, we get $1 - 2(3 - 4s) = 2 + 3s$, so $5s = 7$, so $s = 7/5$; then $t = 3 - 4s = 3 - 28/5 = -13/5$. So $t = -13/5$, $s = 7/5$ is the only solution to the first two equations; plugging it in, we see that the third equation is not satisfied, so the lines do not intersect.

Note that the direction vector of the first line is $v_1 = \langle 1, -2, 3 \rangle$, and the direction vector of the second line is $v_2 = \langle -4, 3, -2 \rangle$. Since as we see, $v_1$ and $v_2$ are not proportional, the lines are not parallel. Thus, the two lines are skew.

**Question 2:** Find an equation of the plane containing the line $L_2$ and parallel to the line $L_1$.

Any plane parallel to $L_1$ has to have a normal vector that is perpendicular to $v_1$. Similarly, if it contains $L_2$, then its normal vector has to be perpendicular to $v_2$. Thus, a normal vector to our plane should be perpendicular to both $v_1$ and $v_2$. To find such a vector, we use the cross product:

$$n = v_1 \times v_2 = \begin{vmatrix}
  i & j & k \\
  1 & -2 & 3 \\
  -4 & 3 & -2
\end{vmatrix} = \langle -5, -10, -5 \rangle.$$

Now, take any point on $L_2$, and use that point and the normal vector we just found to write an equation of this plane. Let us use the point $P = (3, 2, 1)$ that corresponds to $s = 0$. Let’s rescale $n$ by $\frac{1}{5}$ so that it’s easier to deal with; then we get the equation of a plane with the normal vector $w = \frac{1}{5}n = \langle 1, 2, 1 \rangle$ and containing $P$:

$$(x - 3) + 2(y - 2) + (z - 1) = 0.$$
**Question 3:** Find the distance between the lines $L_1$ and $L_2$. To find this distance, all we need to do is find the distance from any point on $L_1$ to the plane from the previous question (imagine the picture, with the plane containing $L_2$ being the floor, and $L_1$ – any line on the ceiling. Since the distance between the floor and the ceiling is always the same, you see that it doesn’t matter which point on $L_1$ we take. Imagine this picture and think about it!)

Take the point on $L_1$ corresponding to $t = 0$ – this is the point $A = (0, 1, 2)$. Now find the distance from the point $(0, 1, 2)$ to the plane $(x - 3) + 2(y - 2) + (z - 1) = 0$. Recall that to do it, we just have to take any point in the plane (we take $P$), and then find the magnitude of the projection (or component) of the vector $\overrightarrow{AP}$ onto $n$ (which is the same as the magnitude of the projection onto $w$). We have: $\overrightarrow{AP} = \langle 3, 1, -1 \rangle$. Then

$$|\text{comp}_w \overrightarrow{AP}| = \frac{|\langle 3, 1, -1 \rangle \cdot \langle 1, 2, 1 \rangle|}{|w|} = \frac{|3 + 2 - 1|}{\sqrt{1 + 2 + 1}} = \frac{4}{\sqrt{6}}.$$

**Answer:** $\frac{4}{\sqrt{6}}$. 