Topics:
- vectors, lines/planes, distances.
- partial derivatives, Chain rule: $W = u(x,y)$, $X = x(s,t)$.
- linear approximations, level curves/surfaces.
- directional derivatives, gradients, hikers.
- max/min critical pts, Lagrange multipliers.

- Integrals: double polar, triple spherical coordinates.
Review of:
- implicit differentiation
- linearization
- distance from point to a plane

Problem:
Let \( z \) be an implicit function defined by the equation
\[
x^2 z^3 + y \sin(5x) = -y^2.
\]

(a) Find the distance from \((4, 5, 6)\) to the tangent plane to this surface at \( P = (1, 1, -1) \).
(b) Find \( \frac{\partial^2 z}{\partial x^2} \), \( \frac{\partial z}{\partial y} \).
Reading the problem:

- We are given a surface.
  (a) We are asked about the tangent plane to this surface at a point P.

  Step 1: find the equation of this tangent plane.
  \[ F(x, y, z) = 0, \] this defines a surface;
  \[ \overrightarrow{DF} \] is \( \perp \) to the tangent plane.

  Here \[ F(x, y, z) = x^2 z^3 + y \sin(5x) + y^2 = 0. \]
  Need \( \overrightarrow{DF} \) at \( P = (1, 1, -1). \)

  Recall: to get an equation of the plane, need a normal vector, will use \( \overrightarrow{DF} \) at \( P \).

\[
\begin{align*}
\frac{\partial F}{\partial x} &= 2xz^3 + 5y \cos(5x) \\
\frac{\partial F}{\partial y} &= 5 \sin(5x) + 2y \\
\frac{\partial F}{\partial z} &= 3x^2 z^2
\end{align*}
\]

Evaluate at \( P \):

- \( \frac{\partial F}{\partial x} \big|_P = -2 + \pi \)
- \( \frac{\partial F}{\partial y} \big|_P = 2 \)
- \( \frac{\partial F}{\partial z} \big|_P = 3 \)

Tangent plane:

\[ (-2 - \pi)(x - 1) + 2(y - 1) + 3(z + 1) = 0. \]
Part (a) asks for distance from \((4, 5, 6)\) to this plane.

Recall: to find distance from a point to a plane:
- pick \(P\) in the plane,
- project \(\overrightarrow{QP}\) onto the normal vector of the plane.
- Take the length of this projection:

\[
= \left| \frac{\overrightarrow{QP} \cdot \overrightarrow{u}}{\|\overrightarrow{u}\|} \right|
\]
- unit vector in the direction of \(\overrightarrow{u}\).

For \(u\): \(\overrightarrow{u} = \frac{\overrightarrow{DF}}{p} = \langle -2, \pi, 2, 3 \rangle\)

\(Q = (4, 5, 6)\) \(\quad P = (1, 1, -1)\)

\(\overrightarrow{QP} = \langle -3, -4, 7 \rangle\)

\[
d = \left| \frac{\overrightarrow{QP} \cdot \overrightarrow{u}}{10} \right| = \left| \frac{-3 \cdot -2 + -4 \cdot \pi + 7 \cdot 3}{\sqrt{(2+\pi)^2 + 9 + 9}} \right|
\]

\[
= \left| \frac{3(2+\pi) - 8 - 21}{\sqrt{(2+\pi)^2 + 12}} \right|
\]
Part (b):
\[
\begin{align*}
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} \\
\frac{\partial z}{\partial y} &= -\frac{F_y}{F_z}
\end{align*}
\]
\[
\begin{align*}
\frac{dx}{dx} &= -\frac{2x^2 + 7y \cos(5x)}{3x^2 + y^2} \\
\frac{dy}{dy} &= -\frac{9x (5x) + 2y}{3x^2 + y^2}
\end{align*}
\]

Distance from point to line

Distance \(d\) from point \(P\) to line \(l\) is:

\[d = \left| \frac{PA \times \vec{u}}{\|\vec{u}\|} \right|\]

\[d = \left| PA - \text{proj}_v PA \right|

\]
We know: \[ d = \| \overrightarrow{PA} - \text{proj}_v \overrightarrow{PA} \| \]

call it \( \overrightarrow{w} \)

What's going on: \[ \overrightarrow{PA} = \overrightarrow{w} + \text{proj}_v \overrightarrow{PA} \]

\( \overrightarrow{w} \) vector parallel to \( v \)

\( \text{proj}_v \overrightarrow{PA} \) vector perpendicular to \( v \).

\( d \) is the length of \( \overrightarrow{w} \)

Consider \[ \overrightarrow{PA} \times \overrightarrow{v} = (\overrightarrow{w} + \overrightarrow{w}) \times \overrightarrow{v} = \overrightarrow{w} \times \overrightarrow{v} + \overrightarrow{w} \times \overrightarrow{v} \]

\( \overrightarrow{w} \times \overrightarrow{v} \) is perpendicular to \( v \).

Now \[ \| \overrightarrow{PA} \times \overrightarrow{v} \| = \| \overrightarrow{w} \times \overrightarrow{v} \| \]

\[ = \| \overrightarrow{w} \| \cdot \| \overrightarrow{v} \| \cdot \sin\left(\frac{\pi}{2}\right) \]

\( \| \overrightarrow{w} \| \perp \overrightarrow{v} \)

\[ = \| \overrightarrow{w} \| \cdot \| \overrightarrow{v} \| . \]

Now: \[ d = \| \overrightarrow{w} \| = \frac{\| \overrightarrow{PA} \times \overrightarrow{v} \|}{\| \overrightarrow{v} \|} . \]
Question 1: $f(x,y,z)$ - a function.
given: at $(1,2,3)$, the direction of the fastest increase of $f$ is $\mathbf{v} = \langle 4, 5, 6 \rangle$

Find an equation of the tangent plane to the level surface of $f(x,y,z)$ at $(1,2,3)$.

Analysis: $f$ - function of 3 variables
so it has level surfaces.
(they are defined by $f(x,y,z) = C$)

given: $f$ - not given.

at $(1,2,3)$, fastest increase of $f$

is in the direction of $\langle 4 , 5 , 6 \rangle$

Know: $\nabla f$ points in the direction of the fastest increase.
so: we are given the direction of \( \nabla f \) at \((4,2,3)\), but not its magnitude.

Is this enough: they are asking for tangent plane to the level surface.

normal: \( \nabla f \).
only direction is sufficient.

Answer: \[ 4(x-1) + 5(y-2) + 6(z-3) = 0 \] tangent plane.

Can I write down the equation of the level surface itself?

No — not enough info given.
but if it was also given that \( f(1,2,3) = 10 \)
can we write the equation of this level surface?

Yes: \( f(x,y,z) = 10 \).
Variation imagine \( f(x, y, z) \) is given:

\[
f(x, y, z) = 3x + 5y - z
\]

Find the equation of the level surface of \( f(x, y, z) \) that contains the point \((1, 2, 3)\).

Level surface = all points where the value of \( f(x, y, z) \) is the same.

if my surface contains \((1, 2, 3)\), then the value is the same as \( f(1, 2, 3) \):

\[
f(1, 2, 3) = 3 \cdot 1 + 5 \cdot 2 - 3 = 10
\]

so our level surface has the equation

\[
3x + 5y - z = 10
\]

\[
f(x, y, z)
\]
Variation, continued:

What if now we have \( f(x, y, z) = 3x + 5y - z \)
and asked to find the equation of the tangent plane
to its level surface at \((1, 2, 3)\)?

For general \( f \), we'll need to compute \( \nabla f \) at \((1, 2, 3)\)
to get a normal vector.

But for this \( f \), the level surface has
the equation \( 3x + 5y - z = 10 \) \( \Rightarrow \) a plane
so its tangent plane is itself:

\[
3x + 5y - z = 10
\]