### Review: Gradients, Level Curves, etc.

<table>
<thead>
<tr>
<th>( f(x, y) )</th>
<th>( f(x, y, z) )</th>
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</thead>
<tbody>
<tr>
<td>( \nabla f = \langle f_x, f_y \rangle ) - a vector in the plane evaluate at your point.</td>
<td>( \nabla f = \langle f_x, f_y, f_z \rangle ) - a vector in ( \mathbb{R}^3 ) (space).</td>
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<tr>
<td><strong>Level Curves:</strong></td>
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<tr>
<td>( f(x, y) = k )</td>
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<tr>
<td>( k = 1, 2, 3, \ldots )</td>
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<tr>
<td>or ( k = a_1, a_2, \ldots )</td>
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<td>(( k ) varies with equal steps).</td>
<td>( )</td>
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<tr>
<td>(They live in the plane - the domain of ( f(x, y) )).</td>
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</tbody>
</table>

\[ \nabla f \text{ at } (a, b) \] is perpendicular to the tangent line to the level curve through \( (a, b) \)

\[ \nabla f = \langle f_x(a, b), f_y(a, b) \rangle \]

\( \nabla f \) points in the direction of the fastest increase of \( f \).

**Level Surfaces**

\( f(x, y, z) = k \) - surface in \( \mathbb{R}^3 \).

For every point in the domain, there is a level surface (for 2 variables, level curve) passing through it.

\[ f(x_0, y_0, z_0) = f(a, b, c) \]

\( \nabla f \) is normal to the tangent plane to the level surface at \( (a, b, c) \).
Directional derivatives

$\mathbf{u}$ - vector \ in \ $\mathbb{R}^2$ \ if \ $f(xy)$ \ \ \ with \ \ 2 \ variables

in \ $\mathbb{R}^3$ \ if \ $f(x,y,z)$ \ \ \ with \ \ 3 \ variables.

$D_{\mathbf{u}}f = \text{rate of change of } f \ \ \text{in the direction of } \mathbf{u}$

(Ex: from homework review:

estimate directional derivative

$\mathbf{v}$ \ at \ $P$, \ the direction

of $f(x,y)$.

of $f(x,y)$.

go in the direction of $\mathbf{v}$

until you meet a level curve.

Get: $D_{\mathbf{v}}f \approx \frac{f(\mathbf{A}) - f(P)}{|A\mathbf{P}|}$

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\[ \nabla f = \overline{\nabla f} \cdot \overline{u} \]

\( \overline{u} \) has to be unit.

(\text{if not unit, rescale}).

because of this:

- special directions:
  1. \( \overline{u} \parallel \overline{\nabla f} \), then
  \[ D_{\overline{u}} f = (|\overline{\nabla f}| \cdot |\overline{u}|) \cdot \overline{u} = |\overline{\nabla f}| \]
    
    \( \overline{u} \) has the largest directional derivative at the given point.
    (at all directions)

  2. \( \overline{u} \perp \overline{\nabla f} \), then
    \[ D_{\overline{u}} f = -|\overline{\nabla f}| \]
    - direction of the fastest decrease.

  3. \( \overline{u} \perp \overline{\nabla f} \), then
    \[ D_{\overline{u}} f = 0 \), so \( \overline{u} \) is tangent to a level curve/level surface.

Caution: for \( f(\mathbf{x}, y) \)

- can use gradient to say things about its graph, but be careful!
Dealing with graphs:
- gradients have 2 components, but if you are on a graph, you are in 3d.

when going "in the direction of the steepest ascent"

- you velocity is a 3d vector
  - its projection to the xy-plane is parallel to $\nabla f(x,y)$;
  - $\vec{V} = \langle v_1, v_2, v_3 \rangle$ then $\langle v_1, v_2 \rangle$ should be in the direction of $\nabla f$.

What about $v_3$?
velocity is tangent to your path on the graph. 
so \( \langle v_1, v_2, v_3 \rangle \) should have correct 'slope': 

\[
\frac{v_3}{|\langle v_1, v_2 \rangle|} \text{ should equal } D_{\hat{u}} f
\]

where \( \hat{u} \) = unit vector in the direction of \( \langle v_1, v_2 \rangle \)

in our situation: \( \hat{u} \uparrow \uparrow \nabla f \)

so: \( \frac{v_3}{|\langle v_1, v_2 \rangle|} = |\nabla f| \)

This allows to find all 3 components.

Example (Dec 2013, #2):

hill \( z = 1000 - 0.02 x^2 - 0.01 y^2 \)

(a) steepest ascent direction at \( (0, 100, 900) \)?

answer: \( \nabla z = \langle -0.04 x, -0.02 y \rangle \)

plug in \( x=0, y=100 \): \( \langle 0, -2 \rangle \)

South.
(b) Slope of the hill \( \mathbf{u} \) at \((0, 100, 900)\) in the direction from (a)? (South)

**Answer:** \(\mathbf{Du} = \mathbf{u}\) - unit vector pointing South: \(<0, -1>\).

\[\mathbf{Du} \cdot \mathbf{u} = \mathbf{Du} \cdot \mathbf{u} = 1 \cdot <0, -2> = <0, -2> \cdot <0, -1> = 2\]

(c) Steepest descent \((N)\) at 5 m/s, what is your rate of change of altitude as you pass through \((0, 100, 900)\)?

**Our velocity:** \(\mathbf{v} = \langle v_1, v_2, v_3 \rangle\)

\[v_2 > 0\]

\[\frac{v_3}{v_2} = -2 \quad \leftarrow \text{slope from (b)}\]

\[\text{given: } (\mathbf{v}) = 5, \quad v_2 + v_3 + 0 = 5^2\]

\[\text{From this, get } v_2, v_3:\]

\[v_2^2 + 4v_2^2 = 25\]

\[v_2 = \sqrt{5}\]

Get: \(\mathbf{v} = \langle 0, \sqrt{5}, -2\sqrt{5} \rangle\)

**Answer:** \(-2\sqrt{5}\) or rate of change of altitude
Triple integrals

- spherical/cylindrical

Usually easier

but for solids made of spheres, cones

centred at (0, 0, 0)

spherical is better.

Example:

E bounded below by \( z = x^2 + y^2 \)

above by \( z = \sqrt{x^2 + y^2} \)

\[ I = \iiint_E z (x^2 + y^2 + z^2) \, dV \]

(a) write \( I \) in terms of cylindrical

(b) spherical

(c) evaluate.
\[ z = \frac{x^2}{y} \]

Projection of \( E \) onto \( xy \)-plane:
\[ R = ? \quad \text{so} \quad R = 1. \]

Find the intersection point:
(Plug \( \lambda = 0 \))
get \((1, 1)\)

Answer:
\[ \iiint \int r \cdot z \left( r^2 + z^2 \right) \, dz \, dr \, d\theta \]

Our surfaces:
- \( z = \frac{x^2}{y} \)
- \( z = r^2 \) in cylindrical
- \( z = \sqrt{x^2 + y^2} \) - cone
- \( z = \sqrt{r^2} = r \)
Spherical:

The cone: \( \psi = \frac{\pi}{4} \)

\[ 0 \leq \theta \leq 2\pi \]

Limits for \( \psi \):

\[ \psi \leq \frac{\pi}{2} \]

(everything is above the xy-plane.
up \( \psi = \frac{\pi}{2} \).

The paraboloid is tangent to the xy-plane, so we do reach \( \psi \).

So:

\[ \frac{\pi}{4} \leq \psi \leq \frac{\pi}{2} \]
Equation of the paraboloid in spherical coordinates:

\[ z = x^2 + y^2 \quad x^2 + y^2 = \rho^2 \sin^2 \varphi \]

Get:
\[ \rho \cos \varphi = \rho^2 \sin^2 \varphi \]
\[ \rho \sin \varphi = \rho \sin^2 \varphi \cdot \cos \varphi \]
\[ \rho \cos \varphi = \rho \sin \varphi \cdot \cos \varphi \cdot \sin \varphi \]

We already have limits for \( \varphi \).
Here we want \( \rho \) in terms of \( \varphi \):
\[ \rho = \frac{\cos \varphi}{\sin^2 \varphi} \]

Answer:
\[ I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{0}^{2^{1/2} \cos \varphi / \sin^2 \varphi} \int_{0}^{2 \cos \varphi \cdot \rho^2 \cdot \rho^2 \sin \varphi} \, dp \, d\rho \, d\varphi \]
Chain Rule

April 2012

\[ \frac{\partial^2 z}{\partial t^2} \]

\[ z = f(x,y) \]

\[ \begin{align*} x &= 2t^2 \\
y &= t^3 \\
f_x(2,1) &= 5 \\
f_y(2,1) &= -2 \\
f_{xx}(2,1) &= 2 \\
f_{xy}(2,1) &= 1 \\
f_{yy}(2,1) &= -4 \end{align*} \]

Find \( \frac{\partial^2 z}{\partial t^2} \) \( t = 1 \).

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**Step 1:**

\[ \frac{dx}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \]

\[ = f_x \cdot 4t + f_y \cdot 3t^2 \]

\( \text{\( \frac{dx}{dt} \) is a function of \( t \).} \)

\( (f_x, f_y \text{ are still functions of } x, y) \)
Step 2: Differentiate again, use product rule.

\[
\frac{d^2 z}{dt^2} = \frac{d}{dt} \left( f_x \cdot 4t + f_y \cdot 3t^2 \right) = \frac{d}{dt} (f_x) \cdot 4t + f_x \cdot 4 \\
+ \frac{d}{dt} (f_y) \cdot 3t^2 + f_y \cdot 6t
\]

\[
= 4t \left( f_{xx} \cdot \frac{dx}{dt} + f_{xy} \cdot \frac{dy}{dt} \right) + f_x \cdot 4 \\
+ 3t^2 \left( f_{yx} \cdot \frac{dx}{dt} + f_{yy} \cdot \frac{dy}{dt} \right) + f_y \cdot 6t
\]

\[
= 4t \left( f_{xx} \cdot 4t + f_{xy} \cdot 3t^2 \right) + 4f_x + 3t^2 \left( f_{xy} \cdot 4t \\
+ f_{yy} \cdot 6t \right)
\]

Now plug in \( t = 1 \).

When \( t = 1 \), \( x = 2 \cdot 1^2 \), \( y = 1^3 \), so \((x,y) = (2,1)\).

So we evaluate \( f \) and \( f_x, f_y, f_{xx}, f_{xy}, f_{yy} \) at \((2,1)\).
Dec 2011 exam #3

bee: along the curve of intersection of the surfaces
\[ 3z + x^2 + y^2 = 2 \quad \text{and} \quad z = x^2 - y^2 \]
in the direction of increase of \( z \).

At \( t = 2 \), bee is at \((1,1,0)\)
speed = 6

(a) Find velocity vector at \( t = 2 \)

(b) \( T \) at \((x,y,z)\) is given by
\[ T = xy - 3x + 2yt + z \]
Find rate of change of \( T \) that the bee is experiencing at \( t = 2 \).

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(a) key point: **velocity** of the bee is tangent to this curve of intersection.
so it lies in the tangent planes of both surfaces.
To find a tangent plane: gradient is its normal vector.

To find the intersection of 2 planes: cross product of their normal vectors is the direction vector of the line of intersection.

So: \[ \nabla f_1 = \langle 2x, 2y, 3 \rangle \] - evaluate at \((1, 1, 0)\)
get: \( \overrightarrow{D_1} = \langle 2, 2, 3 \rangle \)

\[ \nabla f_2 = \langle +2x, -2y, -1 \rangle \] - evaluate at \((1, 1, 0)\)
get: \( \overrightarrow{D_2} = \langle +2, -2, -1 \rangle \).

\[ x^2 - y^2 - z = 0 \] - our second surface

\[ \overrightarrow{D} \] - direction vector of the line

\[ \overrightarrow{D} = \begin{vmatrix} i & j & k \\ 2 & 2 & 3 \\ 1 & -2 & -1 \end{vmatrix} = 4i + 8j - 8k \quad \text{parallel to} \quad \langle 1, 2, -2 \rangle. \]

Find \( \overrightarrow{v} \) - velocity: \( z \) increases, so use \( \langle -1, -2, 2 \rangle \).

- Need to rescale to get magnitude 6.

\[ v = 6 \cdot \frac{\langle -1, -2, 2 \rangle}{\sqrt{1 + 4 + 4}} = \langle -2, 4, 4 \rangle \]

See how these are used (underlined in red) on the next page.
(b) Rate of change of temp:

\[ T(x_t, y_t, z_t) \]

\text{see: } (x(t), y(t), z(t))

\text{chain rule: } \frac{dT}{dt} = \frac{\partial T}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial T}{\partial z} \cdot \frac{dz}{dt} + \frac{\partial T}{\partial t}

= \nabla T \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle + \frac{\partial T}{\partial t}

\text{first 3 components}

\nabla T = \langle -2, 5, 1, 2 \rangle

\[ T(x_t, y_t, z_t) = xy - 3x + 2yt + z \]

\[ T_x = y - 3 \]
\[ T_y = x + 2t \]
\[ T_z = 1 \]
\[ T_t = 2y \]

\text{evaluate at: } x = 1, \ y = 1, \ z = 0, \ t = 2

\text{Answer: } (-2) \cdot (-2) + 5 \cdot (1 - 4)

\[ + 1 \cdot 4 + 2 \]