Quiz 1: vectors; dot product. Section 202.

(1) Find the unit vector in the opposite direction from the vector \( \langle 2, 3 \rangle \).

Answer: \( \frac{1}{\sqrt{13}} \langle 2, 3 \rangle = \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \rangle \).

(2) Let \( P \) be the point \( (0, 3, 0) \). Find the point \( A \) that lies on the \( xz \)-plane, such that the vector \( \overrightarrow{AP} \) is parallel to the vector \( \langle 4, 6, 5 \rangle \).

Solution: Since \( A \) lies on the \( xz \)-plane, it has to have coordinates of the form \( (a, 0, b) \), where we need to find \( a \) and \( b \). Then \( \overrightarrow{AP} = \langle -a, 3, -b \rangle \). We want it to be parallel to \( \langle 4, 6, 5 \rangle \), which means that there exists a scalar \( c \) such that

\[ \langle -a, 3, -b \rangle = c \langle 4, 6, 5 \rangle. \]

Equating components, we get: \(-a = 4c, 3 = 6c, -b = 5c\). From the second equation we find \( c = 1/2 \), and then \( a = -2 \), \( b = -5/2 \). Thus the answer is \( A \) is the point \((-2, 0, -5/2)\).

(3) Suppose the vectors \( \mathbf{v} \) and \( \mathbf{w} \) have the property that

\[ |\text{proj}_w \mathbf{v}| = \frac{1}{2} |\mathbf{v}|. \]

Find the angle between \( \mathbf{v} \) and \( \mathbf{w} \).

Solution: We know the formula for the projection:

\[ \text{proj}_w \mathbf{v} = \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{w}|^2} \mathbf{w}, \]

and therefore taking the magnitude, we get:

\[ |\text{proj}_w \mathbf{v}| = \left| \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{w}|^2} \right| |\mathbf{w}| = \frac{|\mathbf{w} \cdot \mathbf{v}|}{|\mathbf{w}|} = |\mathbf{v}| |\cos(\alpha)|, \]

where \( \alpha \) is the angle between \( \mathbf{v} \) and \( \mathbf{w} \). Thus we get: \( |\mathbf{v}| |\cos(\alpha)| = \frac{1}{2} |\mathbf{v}| \), which happens if and only if \( |\cos(\alpha)| = 1/2 \) (or \( \mathbf{v} = \mathbf{0} \)). Thus if \( \mathbf{v} \neq \mathbf{0} \), we have \( \cos(\alpha) = \pm 1/2 \), so \( \alpha = \pi/3 \) or \( \alpha = 2\pi/3 \).

Answer: \( \alpha = \pi/3 \) or \( \alpha = 2\pi/3 \), if \( \mathbf{v} \neq \mathbf{0} \).