Quiz 1: vectors; dot product. Section 201.

(1) Find the unit vector in the same direction as the vector \( \langle 1, 2, 3 \rangle \).

Answer: \( \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle = \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle \).

(2) Let \( P \) be the point \((0, 1, 2)\). Find the point \( A \) that lies on the \( x \)-axis, such that the vector \( \overrightarrow{AP} \) is perpendicular to the vector \( \langle 4, 6, 5 \rangle \). Hint: use dot product.

Solution: Since \( A \) lies on the \( x \)-axis, it has to have coordinates of the form \((a, 0, 0)\), where we need to find \( a \). Then \( \overrightarrow{AP} = \langle -a, 1, 2 \rangle \). We want it to be perpendicular to \( \langle 4, 6, 5 \rangle \), which means the dot product of these two vectors should be zero:

\[-4a + 6 + 10 = 0.\]

Solving for \( a \), we get \( a = 4 \), so the answer is \( A \) is the point \((4, 0, 0)\).

(3) Suppose the vectors \( \mathbf{v} \) and \( \mathbf{w} \) have the property that

\[ |\text{proj}_w \mathbf{v}| = |\mathbf{v}|. \]

What can you say about \( \mathbf{v} \) and \( \mathbf{w} \)? (Explain your answer).

Solution: The answer is that the vectors \( \mathbf{w} \) and \( \mathbf{v} \) are parallel. There are several possible explanations, all boiling down to the fact that if they were not parallel, projecting onto \( \mathbf{w} \) would make \( \mathbf{v} \) shorter. Here is the formal verification:

We know the formula for the projection:

\[ \text{proj}_w \mathbf{v} = \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{w}|^2} \mathbf{w}, \]

and therefore taking the magnitude, we get:

\[ |\text{proj}_w \mathbf{v}| = \left| \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{w}|^2} \right| |\mathbf{w}| = \frac{|\mathbf{w} \cdot \mathbf{v}|}{|\mathbf{w}|} = |\mathbf{v}| \cdot |\cos(\alpha)|, \]

where \( \alpha \) is the angle between \( \mathbf{v} \) and \( \mathbf{w} \). Clearly, \( |\mathbf{v}| \cdot |\cos(\alpha)| = |\mathbf{v}| \) if and only if \( |\cos(\alpha)| = 1 \) (or \( \mathbf{v} = \mathbf{0} \), which would be parallel to any vector), which means \( \alpha \) is 0 or \( \pi \), i.e. the vectors \( \mathbf{v} \) and \( \mathbf{w} \) are parallel.