Quiz 5, Double integrals. Section 202.

1. (a) (6 marks)
   Sketch the domain of integration and convert the integral to polar coordinates:
   \[ \int_0^2 \int_0^{\sqrt{4-x^2}} x \, dy \, dx. \]

   Answer:
   \[ \int_0^{\pi/2} \int_0^2 f(r \cos \theta, r \sin \theta) r \, dr \, d\theta. \]

(b) (3 marks) Suppose the domain from (a) represents a lamina with the density function \( \rho(x, y) = x \). Find the total mass of the lamina.
   
   Solution: We have:
   \[ M = \int_0^{\pi/2} \int_0^2 (r \cos \theta) r \, dr \, d\theta = \int_0^2 r^3 dr \int_0^{\pi/2} \cos \theta \, d\theta = \frac{8}{3}. \]

(c) (4 marks) With the same density function as in (e), find the \( y \)-coordinate of the centre of mass of this lamina.
   
   Solution: We have:
   \begin{align*}
   \bar{y} &= \frac{1}{M} \int_0^{\pi/2} \int_0^2 (r \sin \theta)(r \cos \theta) r \, dr \, d\theta \\
   &= \frac{3}{8} \int_0^2 r^4 dr \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta = \frac{3}{8} \cdot \frac{2^4}{4} \int_0^{1} u \, du = \frac{6}{8}.
   \end{align*}

2. (7 marks) Consider the integral
   \[ \int_0^9 \int_{3-\sqrt{9-y}}^{3+\sqrt{9-y}} f(x, y) \, dx \, dy. \]
   
   where \( f(x, y) \) is an arbitrary function.
   
   Sketch the domain \( D \) of integration (label all the important features), and switch the order of integration.

   To sketch the domain, we need to graph the functions \( x = 3 \pm \sqrt{9-y} \).
   
   Let us rewrite this equation so that it expresses \( y \) in terms of \( x \) for
easier graphing (which is also useful for switching the order). We get:

\( x - 3 = \pm \sqrt{9 - y} \), so \( y = 9 - (x - 3)^2 = 6x - x^2 \). Then the integral becomes:

\[
\int_0^9 \int_0^{6x-x^2} f(x, y) \, dy \, dx.
\]