This midterm has ?? questions on ?? pages, for a total of ?? points.

Duration: 80 minutes

- Write your name on every page.
- you need to show enough work to justify your answers.
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- Unless a problem states otherwise, you do not have to simplify algebraic expressions to the shortest possible form, and do not have to evaluate long numerical expressions.

Full Name (including all middle names): ________________________________

Student-No: ________________________________________________________

Signature: __________________________________________________________

Section number: _____________________________________________________

Name of the instructor: _______________________________________________
1. Consider the points \(A = (1, 2, 3), B = (1, 5, 1),\) and \(C = (-1, 2, 0),\) respectively.

(a) Find the dot product of the vectors \(\overrightarrow{AB}\) and \(\overrightarrow{AC}\).

(b) Find symmetric equations for the line \(L\) passing through \(A\) and \(B\).

(c) Find the area of the triangle \(ABC\).

(d) Find the angle between the sides \(AB\) and \(AC\) of the triangle \(ABC\). You may express your answer in terms of \(\arccos\).

(e) A boat is travelling East at 10km/hr. To a man on the boat, who measures the wind, it appears that the wind is blowing from the North at 10km/hr. Find the actual direction and speed of the wind.

**Solution:**

(a) \(\overrightarrow{AB} = (1, 5, 1) - (1, 2, 3) = (0, 3, -2)\).
\(\overrightarrow{AC} = (-1, 2, 0) - (1, 2, 3) = (-2, 0, -3)\).
\(\overrightarrow{AB} \cdot \overrightarrow{AC} = (0, 3, -2) \cdot (-2, 0, -3) = 0 + 0 + 6 = 6\).

(b) The line \(L\) passing through \(A\) and \(B\) is:

\[
\begin{align*}
x &= 1 \\
y &= 2 + 3t \\
z &= 3 - 2t
\end{align*}
\]

The symmetric equation is:

\[
\frac{y - 2}{3} = \frac{z - 3}{-2}, \quad x = 1.
\]

(c) The area of the triangle \(ABC\) is:

\[
\frac{1}{2} |\overrightarrow{AB}| |\overrightarrow{AC}| \sin \theta = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \left| \begin{array}{ccc} i & j & k \\ 0 & 3 & -2 \\ -2 & 0 & -3 \end{array} \right| = \frac{1}{2} |(-9, 4, 6)| = \frac{1}{2} \sqrt{133}.
\]

(d) \(\theta = \arccos \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \arccos \frac{6}{\sqrt{9 + 4 \sqrt{4 + 9}} = \arccos \frac{6}{13}}\)

(e) The velocity of the boat is \((\sqrt{10}, 0)\);

The velocity of the wind is \((w_1, w_2)\);

The velocity of feels of wind is \((0, -\sqrt{10})\);
$(\sqrt{10}, 0) + (w_1, w_2) = (0, -\sqrt{10})$.

∴ $(w_1, w_2) = (0, -\sqrt{10}) - (\sqrt{10}, 0) = (-\sqrt{10}, -\sqrt{10})$.

$|(w_1, w_2)| = \sqrt{10 + 10} = 2\sqrt{5}$.

The wind is traveling in the direction of SW and at the speed of $2\sqrt{5} km/h$. 
2. Consider the point \( A = (4, 1, 3) \) and the plane \( P \) given by the equation \( x + 2y - 3z = 2 \).

(a) Find the plane which passes through the point \( A \) and is parallel to the plane \( P \).

(b) Find the distance between the two parallel planes from part (a).

(c) Find the parametric equation of the line \( L \) of intersection of the plane \( P \) with the \( xz \)-plane.

(d) Find the distance from the point \( B \) with the coordinates \((0, 3, -4)\) to the line \( L \) from Part (c) above.

**Solution:**

(a) Since the plane is parallel to \( P \), the norm of the plane is parallel to \((1, 2, -3)\).

\[
\therefore \text{For every } Q = (x, y, z) \text{ on the plane,}

(x - 4, y - 1, z - 3) \cdot (1, 2, -3) = 0
\]

\[
x - 4 + 2(y - 1) - 3(z - 3) = 0
\]

\[
x + 2y - 3z = -3
\]

The equation of the plane is \( x + 2y - 3z = -3 \).

(b) The distance between the two planes is the same as the distance between \( A \) and \( P \). It is easy to check that the point \( M = (0, 1, 0) \) is on the plane \( P \).

\[
\overrightarrow{AM} = (0, 1, 0) - (4, 1, 3) = (-4, 0, -3).
\]

Distance \( d = \left| \frac{\overrightarrow{AM} \cdot (1, 2, 3)}{|(1, 2, 3)|} \right| = \left| \frac{(-4, 0, -3) \cdot (1, 2, -3)}{\sqrt{1 + 4 + 9}} \right| = \frac{5\sqrt{14}}{14}.
\]

(c) Need to find a point lies on \( L \). Since \( P \) intersect with the \( xz \)-plane, every point on \( L \) satisfies \( y = 0 \).

\[\therefore x - 3z = -3.\]

Let \( z = 0 \), \( \Rightarrow x = -3 \).

\(\therefore \) The point \((-3, 0, 0)\) is on \( L \).

The direction vector of \( L \) is perpendicular to \((0, 1, 0)\) and \((1, 2, -3)\).

The direction vector of \( L = (0, 1, 0) \times (1, 2, -3) = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 2 & -3 \end{vmatrix} = (-3, 0, -1)\).

The parametric equation of \( L \) is

\[
\begin{cases}
x = -3 - 3t \\
y = 0 \\
z = -t
\end{cases}
\]

(d) \( B = (0, 3, -4) \), Let \( Q = (x, y, z) \) be a point on \( L \).

\[\therefore Q = (-3 - 3t, 0, -t) \text{ for some } t \in \mathbb{R}.\]

\[\therefore \overrightarrow{BQ} = (-3 - 3t, 0, -t) - (0, 3, -4) = (-3 - 3t, -3, -t + 4)\]
\[ \therefore \overrightarrow{BQ} \perp \overrightarrow{L}, \]
\[ \therefore (\mathbf{−3}−3t,−3,−t+4) \cdot (−3,0,−1) = 0. \]
\[ \therefore 9 + 3t + t - 4 = 0 \Rightarrow t = -\frac{5}{4}. \]
\[ \therefore \overrightarrow{BQ} = \left(\frac{3}{4}, -3, \frac{21}{4}\right). \]
\[ \text{distance} = |\overrightarrow{BQ}| = \sqrt{\left(\frac{3}{4}\right)^2 + (-3)^2 + \left(\frac{21}{4}\right)^2} = \frac{3}{4} \sqrt{66}. \]
3. Consider the function \( g(x, y) = \sqrt{4 - x^2 - 2y^2} \).

(a) Determine the domain and range of the function \( g \).

(b) Sketch the level curves \( g(x, y) = k \) for the constants \( k = 0, 1, 2 \).

\[ \text{Solution:} \]

(a) Domain:
\[ 4 - x^2 - 2y^2 \geq 0. \]
\[ \therefore x^2 + 2y^2 \leq 4. \]
Domain = \{ \( (x, y) \in \mathbb{R}^2 | x^2 + 2y^2 \leq 4 \) \}.

Range:
\[ \therefore x^2 \geq 0, 2y^2 \geq 0, \]
\[ \therefore x^2 + 2y^2 \geq 0. \]
\[ \therefore 4 - x^2 - 2y^2 \leq 4. \]
\[ \therefore \text{Range} (g) = [0, 2]. \]

(b) Case 1: \( k = 0 \)
\[ g(x, y) = \sqrt{4 - x^2 - 2y^2} = 0 \]
\[ 4 - x^2 - 2y^2 = 0 \]
\[ x^2 + 2y^2 = 4 \]
\[ \frac{x^2}{4} + \frac{y^2}{2} = 1 \]

\[ (0, \sqrt{2}) \]
\[ (0, -\sqrt{2}) \]
\[ (-2, 0) \]
\[ (2, 0) \]

Case 2: \( k = 1 \)
\[ g(x, y) = \sqrt{4 - x^2 - 2y^2} = 1 \]
\[ 4 - x^2 - 2y^2 = 1 \]
\[ x^2 + 2y^2 = 3 \]
\[ \frac{x^2}{3} + \frac{y^2}{3/2} = 1 \]
Case 2: $k = 2$

$g(x, y) = \sqrt{4 - x^2 - 2y^2} = 2$

$4 - x^2 - 2y^2 = 4$

$x^2 + 2y^2 = 0$

$x = 0$ and $y = 0$
4. Let \( u(x, y) = F(x^2 + ay^2) \) for some arbitrary differentiable function \( F \), where \( a > 0 \) is a constant.

(a) If \( F(t) = \ln(t) \), sketch the level curves (the contour plot) of the function \( u(x, y) \) in the three cases: \( a > 0 \), \( a = 0 \), \( a < 0 \).

(b) If \( F(t) = \ln(t) \), find \( u_x \) and \( u_y \).

(c) Let the function \( F \) be arbitrary. Find the value of \( a \) such that

\[
5yu_x = xu_y.
\]

**Solution:**

(a) Case 1: \( a > 0 \)

\[
u(x, y) = F(x^2 + ay^2)
\]

\[
u(x, y) = \ln(x^2 + ay^2) = k \text{ for some } k \in \mathbb{R}.
\]

\[
x^2 + ay^2 = e^k
\]

\[
\frac{x^2}{e^k} + \frac{y^2}{e^k/a} = 1.
\]

Case 2: \( a = 0 \)

\[
u(x, y) = \ln(x^2) = k \text{ for some } k \in \mathbb{R}.
\]

\[
x^2 = e^k
\]

\[
x = \pm e^{k/2}.
\]
Case 3: $a < 0$

$u(x, y) = F(x^2 + ay^2)$

$u(x, y) = \ln(x^2 + ay^2) = k$ for some $k \in \mathbb{R}$.

$x^2 + ay^2 = e^k$

$\frac{x^2}{e^k} + \frac{y^2}{e^k/a} = 1.$

(b) $u(x, y) = \ln(x^2 + ay^2)$

$u_x = \frac{2x}{x^2 + ay^2},$

$u_y = \frac{2ay}{x^2 + ay^2}.$

(c) $u_x = F'(x^2 + ay^2) \cdot 2x,$

$u_y = F'(x^2 + ay^2) \cdot 2ay.$

$\therefore 5yu_x = xu_y$

$\therefore 10xyF'(x^2 + ay^2) = 2axyF'(x^2 + ay^2).$

If $F' \neq 0$, $2a = 10$. Hence $a = 5.$
If $F' = 0$, then $a$ can be any real number.
5. (a) Let \((x, y, 0)\) be an arbitrary point in the \(xy\)-plane. What is the distance from this point to the plane \(z = 4\)?

(b) In \(\mathbb{R}^2\), find an equation for, and draw the set of points that are equidistant from the point \((0, 0)\) and the line \(y = 2\).

(c) Find the equation of the surface \(S\) consisting of all points in \(\mathbb{R}^3\) that are equidistant from the point \((0, 0, 0)\) and the plane \(z = 4\).

(d) Classify the surface \(S\) from part (c) as an ellipsoid, a paraboloid, or a hyperboloid of either 1 or 2 sheets.

**Solution:**

(a) Any point on the plane \(z = 4\) can be written as \((x, y, 4)\).

The distance \(d\) between \((x, y, 0)\) and \(z = 4\) is:

\[
d = |((x, y, 4) - (x, y, 0)) \cdot (0, 0, 1)| = |(0, 0, 4) \cdot (0, 0, 1)| = 4.
\]

(b) Let \(P = (x, y)\) be a point that is equidistant from the point \((0, 0)\) and the line \(y = 2\).

\[\therefore \sqrt{x^2 + y^2} = |y - 2|.
\]

\[x^2 + y^2 = y^2 - 4y + 4\]

\[y = 1 - \frac{x^2}{4}\]

(c) Let \(Q = (x, y, z)\) be a point that is equidistant from the point \((0, 0, 0)\) and the line \(z = 4\).

\[\therefore \sqrt{x^2 + y^2 + z^2} = |z - 4|\]

\[x^2 + y^2 + z^2 = z^2 - 8z + 16\]

\[S : x^2 + y^2 + 8z = 16.\]

(d) For fixed \(z = z_0 < 2\), the level curve on the plane \(z = z_0\) is a circle.

For fixed \(x = x_0\), the level curve on the plane \(x = x_0\) is a parabola.

For fixed \(y = y_0\), the level curve on the plane \(y = y_0\) is a parabola.

Therefore, the surface \(S\) is a paraboloid of one sheet.