1. (a) Find a unit vector perpendicular to \( \langle 2, 3 \rangle \).

\[
\text{Answer: } \frac{\langle -3, 2 \rangle}{\sqrt{13}} = \langle \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle
\]

(b) Find the volume of the parallelepiped defined by the vectors \( \langle 1, 2, 3 \rangle \), \( \langle 1, 0, 1 \rangle \) and \( \langle 0, 2, 5 \rangle \).

\[
\text{Answer: } \det \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix} = | -6 | = 6
\]

(Note: This determinant is the scalar triple product \( \langle 1, 2, 3 \rangle \cdot (\langle 1, 0, 1 \rangle \times \langle 0, 2, 5 \rangle) \).)
2. Consider the lines given in parametric form by:

\[ L_1 : \mathbf{r}_1(t) = (3, 7, 1) + t(1, 2, 0) \] and \[ L_2 : \mathbf{r}_2(t) = (-1, 2, 0) + t(2, 1, 1). \]

(a) Write \( L_1 \) in symmetric form.

\[
\begin{align*}
\text{Answer:} \\
\frac{x-3}{1} &= \frac{y-7}{2} \\
 2 &= 1
\end{align*}
\]

(b) Find the intersection of \( L_1 \) and \( L_2 \).

\[
\begin{align*}
\mathbf{r}_1(t) &= 3 + t \\
\mathbf{r}_2(s) &= -1 + 2s \\
\mathbf{y}_1(t) &= 7 + 2t \\
\mathbf{y}_2(s) &= 2 + s \\
\mathbf{z}_1(t) &= 1 \\
\mathbf{z}_2(s) &= s
\end{align*}
\]

We get:

\[
\begin{align*}
3 + t &= -1 + 2s \\
7 + 2t &= 2 + s \\
1 &= s
\end{align*}
\]

\( \Rightarrow \) \( \begin{align*} 3 + t &= 1 \\
7 + 2t &= 3
\end{align*} \) \( \Rightarrow \) \( t = -2 \)

The point \( (1, 3, 1) \) is the intersection point.

It corresponds to \( t = -2 \) on the first line,

\( s = 1 \) on the second line.

(c) Find the plane containing both \( L_1 \) and \( L_2 \).

\( \mathbf{n} \) - normal vector for this plane.

\[
\mathbf{n} = \langle 1, 2, 0 \rangle \times \langle 2, 1, 1 \rangle = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{vmatrix} = 2\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}
\]

Use the point from (b).

\[
\text{Answer:} \quad 2(x-1) - (y-3) - 3(z-1) = 0.
\]
3. Find the equation for all the points $P$ satisfying that the distance from $P$ to $(1, 1, -1)$ is twice of the distance from $P$ to the plane $z = -1/4$. Sketch (as a 3d surface) and name this surface.

Let $P = (x, y, z)$. We have:

$$\sqrt{(x-1)^2 + (y-1)^2 + (z+1)^2} = 2 \left| z + \frac{1}{4} \right|$$

$$(x-1)^2 + (y-1)^2 + (z+1)^2 = 4 \left( z^2 + \frac{1}{2}z + \frac{1}{16} \right)$$

$$(x-1)^2 + (y-1)^2 + (z+1)^2 = 4z^2 + 2z + \frac{1}{4}$$

$$(x-1)^2 + (y-1)^2 + z^2 - 2z = 4z^2 - 2z = \frac{1}{4} - 1$$

$$(x-1)^2 + (y-1)^2 - 3z^2 = -\frac{3}{4}$$

Multiply the equation by $-\frac{4}{3}$ to set it to the standard form:

$$4z^2 - \frac{y}{3}(x-1)^2 - \frac{y}{3}(y-1)^2 = 1$$

- Hyperboloid of 2 sheets
  (axis of symmetry: the vertical line through the point $(1, 1)$)
4. Let \( f(x, y) = xe^{xy} + y^2x + x \). All parts of this question refer to this function.

(a) Find the partial derivatives \( f_{xx}, f_{xy}, \) and \( f_{yy} \). (Show all your work!)

\[
\begin{align*}
    f_x &= e^{xy} + xy e^{xy} + y^2 + 1, \\
    f_y &= x^2 e^{xy} + 2xy \\
    f_{xx} &= ye^{xy} + xy^2 e^{xy} + y e^{xy} \\
    f_{xy} &= xe^{xy} + xe^{xy} + y x^2 e^{xy} + 2y = 2xe^{xy} + x^2 ye^{xy} + 2y \\
    f_{yy} &= x^3 e^{xy} + 2x
\end{align*}
\]

(b) Find the equation of the tangent plane to the graph \( z = f(x, y) \) at the point \((1, 0, e^{1+1})\).

\[
2 = f(1, 0)
\]

\[
z = 2 + f_x(1, 0)(x-1) + f_y(1, 0)(y-0)
\]

We have:
\[
\begin{align*}
    f_x(1, 0) &= 1 + 1 = 2 \\
    f_y(1, 0) &= e^1 = 1
\end{align*}
\]

Answer:
\[
z = 2 + 2(x-1) + y
\]
Recall from last page: \( f(x, y) = xe^{xy} + y^2x + x \).

(c) At \((1, 0, e^{2^2})\), what is the slope of the tangent line to the intersection of the graph \( z = f(x, y) \) with the \(xz\)-plane (we will call it the slope in the \(x\)-direction)?

\[ \text{This is the plane } y=0, \text{ so the slope is } f_x(1,0), \text{ which is } 2 \text{ as evaluated in } (b) \]

Answer: \(2\)

(d) What does \(f_{xy}(1, 0)\) tell us about how the slope of the tangent line in the \(x\)-direction is changing around \((0, 1)\)?

\[ f_{xy}(1,0) = 2 > 0 \]

\(f_{xy}\) is the rate of change of \(f_x\) as \(y\) varies. So: \(f_{xy}(1,0) = 2\) means that the slope of the graph of \(f\) in the \(x\)-direction increases as \(y\) increases (around \((1,0)\)).
5. Consider the surface defined by the equation \( F(x, y, z) = \ln(x+2y+3z) - 2x - y - 2z = 0 \).

(a) Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \):

Use implicit differentiation.

\[
F_x = \frac{1}{x+2y+3z} - 2
\]

\[
F_y = \frac{2}{x+2y+3z} - 1
\]

\[
F_z = \frac{3}{x+2y+3z} - 2
\]

\[
\frac{\partial z}{\partial x} = \frac{-1-2(x+2y+3z)}{3-2(x+2y+3z)}
\]

\[
\frac{\partial z}{\partial y} = \frac{-2-2(x+2y+3z)}{3-2(x+2y+3z)}
\]

2 marks

(b) Find the total differential of \( z \) as a function of \( x \) and \( y \) at the point \((0, 2, -1)\):

Evaluate \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) at \((0, 2, -1)\):

\[
\frac{\partial z}{\partial x} \bigg|_{(0, 2, -1)} = -\frac{1}{1} = 1
\]

\[
\frac{\partial z}{\partial y} \bigg|_{(0, 2, -1)} = -\frac{2}{3-2} = -1
\]

Then the total differential is

\[
dz = dx - dy
\]

3 marks

(c) On this surface and near the point \((0, 2, -1)\), find an approximate value of \( z \) when \( x = 0.1 \), \( y = 1.9 \).

\[
z(0.1, 1.9) = z(0, 2) + \frac{\partial z}{\partial x}(0, 2) \cdot 0.1 + \frac{\partial z}{\partial y}(0, 2) \cdot (-0.1)
\]

\[
= -1 + 0.1 + 0.1 = -0.8
\]
6. (a) Let \( f(u, v) = e^u \cos v \), and let \( u(x, y) = ax + by \), \( v(x, y) = -bx + ay \), where \( a \) and \( b \) are real numbers. Find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial^2 f}{\partial x^2} \).

\[
\begin{align*}
    f_u &= e^u \cos v, \\
    f_v &= -e^u \sin v; \\
    \frac{\partial u}{\partial x} &= a, \\
    \frac{\partial v}{\partial x} &= -b.
\end{align*}
\]

We have, from chain rule:

\[
\frac{\partial f}{\partial x} = f_u \cdot \frac{\partial u}{\partial x} + f_v \cdot \frac{\partial v}{\partial x} = e^u \cos v \cdot a + e^u \sin v \cdot (-b)
\]

\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( a e^u \cos v + b e^u \sin v \right)
\]

chain rule again!

\[
= a \left( a e^u \cos v + b e^u \sin v \right) + b \left( a e^u \sin v + (-b) e^u \cos v \right)
\]

(b) Suppose you are given that a certain function \( f(u, v) \) satisfies the differential equation \( f_{uu} + f_{uv} = f \). Show that if you let \( g(x, y) = f(ax + by, -bx + ay) \), then \( g(x, y) \) will satisfy the equation

\[
g_{xx} + g_{yy} = (a^2 + b^2)g.
\]

Here we do the same thing but with the abstract function \( f(u, v) \):

\[
g_{xx} = \frac{\partial^2 g}{\partial x^2} \left( f(ax + by, -bx + ay) \right)
\]

(let \( u = ax + by \))

\( v = -bx + ay \)

First compute the first derivative:

\[
g_x = f_u \cdot \frac{\partial u}{\partial x} + f_v \cdot \frac{\partial v}{\partial x} = a f_u - b f_v
\]

chain rule

\[
g_{xx} = \frac{\partial}{\partial x} \left( a f_u - b f_v \right) = a \left( \frac{\partial f_u}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f_v}{\partial u} \cdot \frac{\partial u}{\partial x} \right)
\]

chain rule again, an above

\[
- b \left( \frac{\partial f_u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f_v}{\partial v} \cdot \frac{\partial v}{\partial x} \right)
\]

\[
= a \left( f_{uu} \cdot a - b f_{uv} \right) - b \left( f_{uv} \cdot a - f_{vu} \cdot b \right)
\]

\[
= a^2 f_{uu} - ab f_{uv} - ab f_{vu} + b^2 f_{vv}
\]

\[
= a^2 f_{uu} - 2ab f_{uv} + b^2 f_{vv}
\]

Here note that we are assuming \( f_{uv} = f_{vu} \) since \( f \) is smooth.
\[ g_{yy} = \frac{\partial}{\partial y} (b f_{uu} + a f_{uv}) \]
\[ = b \left( \frac{\partial f_{uu}}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f_{uu}}{\partial v} \cdot \frac{\partial v}{\partial y} \right) \]
\[ + a \left( \frac{\partial f_{uv}}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f_{uv}}{\partial v} \cdot \frac{\partial v}{\partial y} \right) \]
\[ = b \left( f_{uu} \cdot (u') + f_{uv} \cdot a \right) \]
\[ + a \left( f_{uv} \cdot b + f_{vv} \cdot a \right) \]
\[ = b^2 f_{uu} + 2ab f_{uv} + a^2 f_{vv}. \]

Again, note \( f_{uv} = f_{vu} \)

Finally, putting it together:

\[ g_{xx} + g_{yy} = (a^2 f_{uu} - 2ab f_{uv} + b^2 f_{vv}) \]
\[ + (b^2 f_{uu} + 2ab f_{uv} + a^2 f_{vv}) \]
\[ = (a^2 + b^2) (f_{uu} + f_{vv}) = (a^2 + b^2) f(u,v) \]

"as given, \( f(u,v) \) is the problem

\[ = (a^2 + b^2) g(x,y), \text{ as required} \]

because \( f(u,v) = g(x,y) \) by definition."