Math 200 Midterm II (November 1, 2012)
Sections 107. Instructor: Julia Gordon

Name:
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Problem 1:
(a) [2 points] Let \( F(x, y, z) = \sin(xz + y) - x^2 + z \). Find the expression for \( \nabla F \) - the gradient of \( F \) at a point \((x, y, z)\).

\[
\nabla F = \left< \cos(xz + y) \cdot z - 2x, \cos(xz + y), \cos(xz + y) \cdot x + 1 \right>
\]

(b) [3 points] For the same function \( F(x, y, z) \), find the directional derivative \( D_{\mathbf{u}} F \) at the point \((1, \pi/2, 0)\) in the direction of the vector \((1, 2, 3)\).

The unit vector in the direction of \( \langle 1, 2, 3 \rangle \) is \( \mathbf{u} = \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle \).

\[
\nabla F|_{(1, \pi/2, 0)} = \langle -2, 0, 1 \rangle \quad D_{\mathbf{u}} F = \langle -2, 0, 1 \rangle \cdot \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle
\]

(compute: \( \cos(1 \cdot 0 + \pi/2) = 0 \) )

\[
= -\frac{2}{\sqrt{14}} + \frac{3}{\sqrt{14}} = \frac{1}{\sqrt{14}}
\]

(c) [4 points] Find the equation of the tangent plane to the surface defined by the equation \( x^2 - z = \sin(xz + y) \) at the point \((1, \pi/2, 0)\).

This surface is the level surface of \( F(x, y, z) = \sin(xz + y) - x^2 + z \).

Then, \( \nabla F|_{(1, \pi/2, 0)} \) is normal to the tangent plane.

Answer: \[-2(x-1) + 0 \cdot (y-\pi/2) + 1 \cdot (z-0) = 0\]
Problem 2: The temperature $T$ at a point on a metal plate depends on the coordinates $x,y$. We do not know what the temperature function is, but the following information is given: at the point $P(10,11)$, the temperature does not change in the direction of the vector $i + j$; and the rate of change of the temperature at $P$ in the direction of the vector $i - j$ equals $-3\sqrt{2}$ degrees per centimeter.

(a) [2 points] If an ant is crawling through the point $P$ in the direction of the vector $-i - j$ at the speed of 4 cm/s, what is the rate of change of temperature that the ant is experiencing?

(b) [4 points] Find the gradient of the temperature function at the point $P$.

(c) [3 points] If a beetle is crawling through the point $P$ at the speed of 4 cm/s in the direction of the vector $i + 2j$, what is the rate of change of temperature that the beetle is experiencing?

(a) \[ \mathbf{i} + \mathbf{j} \quad \text{we are given that} \quad \nabla T = 0 \quad \text{where} \quad \mathbf{a} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \]

\[ \text{(the unit vector in the direction of (i+j)} \]

\[ \text{Then} \quad \nabla T = -\nabla T = \mathbf{0} \]

(b) \[ \nabla T \bigg|_P = \langle a, b \rangle \quad \text{we are given:} \]

\[ \nabla T = 0 \quad \text{where} \quad \mathbf{u}_1 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \]

\[ \nabla T = -3\sqrt{2} \quad \text{where} \quad \mathbf{u}_2 = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \]

\[ \text{Then} \]

\[ \begin{bmatrix} \langle a, b \rangle \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = 0 \\ \langle a, b \rangle \cdot \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = -3\sqrt{2} \end{bmatrix} \]

\[ \begin{cases} 
    a + b = 0 \\
    a - b = 6 
\end{cases} \]

\[ \begin{cases} 
    a = 3 \\
    b = -3 
\end{cases} \]

\[ \nabla T \bigg|_P = \langle 3, -3 \rangle \]
2 (c) The velocity vector of the beetle is \( 4 \hat{u} \), where \( \hat{u} \) is the unit vector in the direction \( i + 2j \):
\[
\hat{u} = \left< \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right>.
\]
Suppose \( x(t), y(t) \) are coordinates of the beetle.

\[
\frac{dT}{dt} = \frac{\partial T}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt} = \hat{v} \cdot \left< \frac{dx}{dt}, \frac{dy}{dt} \right>
\]

\[
= \left< 3, -3 \right> \cdot \left< \frac{4}{\sqrt{5}}, \frac{2y}{\sqrt{5}} \right>
\]

\[
= \frac{12}{\sqrt{5}} - \frac{2y}{\sqrt{5}} = \frac{-12}{\sqrt{5}} \text{ m/s}\]
Problem 3: A hiker is going down a hill whose shape is given by \( z = e^{x^2 - y^2} \).

(a) [2 points] Find the direction of the steepest descent when the hiker is at the point \( P(1, 1/2, e^{2.5}) \). State your answer in terms of the directions of the compass (N, S, W, E, NW, etc.); you can assume that East is the positive direction of the \( x \)-axis, and North is the positive direction of the \( y \)-axis.

(b) [3 points] Suppose that the trail the hiker is on follows the path of the steepest descent from \( P \). Find the angle the trail is descending at (compared to the horizontal plane).

(c) [3 points] For the same trail as in (b), find a tangent vector to the trail at \( P \). (It should be a vector with three components).

(a) \( \bar{\nabla}f = \langle -2xe^{y-x^2-y^2}, -ye^{y-x^2-y^2} \rangle \)

(where \( f(x,y) = e^{y-x^2-y^2} \) is the graph of this function).

\( \bar{\nabla}f \mid_{(1,1/2)} = \langle -2e^{1/2}, -2e^{2.5} \rangle \)

This vector points SW, the steepest descent is opposite to the gradient, so it'll be \( \text{NE} \).

(b) Let \( \alpha \) be the angle of the trail.

\[ \tan(\alpha) = \frac{\bar{\nabla}f}{|\bar{\nabla}f|}, \text{ where } \bar{u} \text{ is the unit vector defining the direction of the trail.} \]

In our case, \( \bar{u} \) is opposite to \( \bar{\nabla}f \), so

\[ \bar{\nabla}f = -1 \bar{\nabla}f, \quad |\bar{\nabla}f| = -\sqrt{4(e^{2.5})^2 + 4(e^{2.5})^2} = -2e^{2.5}\sqrt{2} \]

So, \( \alpha = \tan^{-1}(-2e^{2.5}\sqrt{2}) \).

Note that \( \alpha \) being negative indicates that the hiker is going down.

(c) This vector should have the projection onto the \( xy \)-plane that is opposite to \( \bar{\nabla}f \), and "slope" as in (b).

At the unit vector \( \bar{u} \) in the direction of \(-\bar{\nabla}f\)

We get: \( \bar{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -2e^{2.5}\sqrt{2} \rangle \)
Problem 4: [5 points]

A function \( f(x, y) \) satisfies: \( \frac{\partial f}{\partial x}(x, y) = a \) and \( \frac{\partial f}{\partial y}(x, y) = 3 \). The variables \( x, y, s, \) and \( w \) satisfy the relations:

\[
\begin{align*}
sc^{x+w} &= 10 \\
2s^2 + w^2 &= y.
\end{align*}
\]

Find the value of \( a \) such that \( \frac{\partial f}{\partial s} \) is zero at the point \( s = 10, w = 0 \).

we have (from Chain Rule):

\[
\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}
\]

Note that when \( (s, w) = (10, 0) \), we have

\[
(x, y) = (0, 2), \quad \text{because:}
\]

\[
y = 2 \cdot 10^2 + w^2 = 200
\]

\[
10e^{x+0} = 10, \quad \text{so} \quad e^x = 1 \quad \text{and} \quad x = 0.
\]

Then

\[
\left. \frac{\partial f}{\partial s} \right|_{(10, 0)} = \frac{\partial f}{\partial x} \left|_{(0, 200)} \right. \cdot \left. \frac{\partial x}{\partial s} \right|_{(10, 0)} + \frac{\partial f}{\partial y} \left|_{(0, 200)} \right. \cdot \left. \frac{\partial y}{\partial s} \right|_{(10, 0)}
\]

\[
= a \left. \frac{\partial x}{\partial s} \right|_{(10, 0)} + 3 \left. \frac{\partial y}{\partial s} \right|_{(10, 0)}
\]

\[
\frac{\partial y}{\partial s} = \frac{\partial}{\partial s} (2s^2 + w^2) = 4s, \quad \text{so} \quad \left. \frac{\partial y}{\partial s} \right|_{(10, 0)} = 40
\]

It remains to find \( \frac{\partial x}{\partial s} \). For that we need implicit differentiation, since \( x \) is defined implicitly.

Differentiate \( se^{x+w} = 10 \) with respect to \( s \).

Get:

\[
\left. \frac{\partial}{\partial s} (se^{x+w}) = 0. \quad \text{Use product rule (treat \( x \) as a function of \( s \))}
\right.
\]

\[
e^{x+w} + se^{x+w} \frac{\partial x}{\partial s} = 0
\]

\[
\frac{\partial x}{\partial s} = -\frac{e^{x+w}}{se^{x+w}} = -\frac{1}{s}.
\]

\[
\left. \frac{\partial x}{\partial s} \right|_{(10, 0)} = -\frac{1}{10}
\]
Putting it all together, get:

$$\frac{\partial f}{\partial s}\bigg|_{(10, 10)} = a \cdot \left(-\frac{1}{10}\right) + 3.40 = 120 - \frac{a}{10}$$

We want: \(120 - \frac{a}{10} = 0\), so \(a = 1200\).
Problem 5: All parts of this problem are about the function \( f(x,y) = x^2 + y^3 - y^2 x - y^2 \).

(a) [4 points] Find all critical points of \( f \).
(b) [4 points] Classify the critical points of \( f \) using the second derivative test.
(c) [6 points] Find the list of points where you need to compare the values of \( f \) in order to find its absolute minimum on the closed bounded domain bounded by the curve \( x = y^2/4 \) and the vertical line \( x = 1 \).

\[
\begin{align*}
&f_x = 2x - y^2 \\
&f_y = 3y^2 - 2xy - 2y
\end{align*}
\]

\[
\begin{cases}
2x - y^2 = 0 \\
3y^2 - 2xy - 2y = 0
\end{cases}
\]

If \( y = 0 \), we get \( x = 0 \).
If \( y \neq 0 \), we have \( y^2 = 2x = 3y - 2 \).

\[
y^2 - 3y + 2 = 0 \quad y_{1,2} = 1 \text{ and } 2
\]

\[
x = \frac{y^2}{2}
\]

Get: \( (0,0) \) and \( (2,2) \)

Answer: \( (0,0) \) \( (\frac{1}{2},1) \) \( (2,2) \)

(b) \( f_{xx} = 2 \)
\( f_{xy} = -2y \)
\( f_{yy} = 6y - 2x - 2 \)

at \( (0,0) \):
\[
D = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4
\]

(0,0) is a saddle point

at \( (\frac{1}{2},1) \):
\[
D = \begin{vmatrix} 2 & -2 \\ -2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 0 & 3 \end{vmatrix} = 6 - 4 > 0 \\
f_{xx} > 0
\]

\( (\frac{1}{2},1) \) is a local min
The point \((2, 2)\):

\[
D = \begin{vmatrix}
-4 & -2 \\
-2 & -2
\end{vmatrix}
\begin{vmatrix}
-4 & 6 \\
-2 & -2
\end{vmatrix}
= \begin{vmatrix}
2 & -4 \\
-4 & 6
\end{vmatrix} = 12 - 16 < 0,
\]
so it is a saddle point.

Inside the domain, we have the critical pt \((\frac{1}{2}, 1)\); also, \((0, 0)\) happens to be on the boundary.

Now we have to look for possible abs min points on the boundary.

1) The line \(x = 1\), \(-2 \leq y \leq 2\).

Plug in \(x = 1\) into \(f(x, y)\), get:

\[
f(1, y) = 1 + y^3 - y^2 - y^2 = 1 + y^3 - 2y^2
\]

\[
f'(by) = 3y^2 - 4y
\]

Get the points \((1, 0)\), and \((1, \frac{1}{3})\)

2) \(x = \sqrt[4]{y}\). Plug this into \(f(x, y)\).

Get:

\[
f\left(\frac{y^2}{4}, y\right) = \left(\frac{y^2}{4}\right)^2 + y^3 - y^2 \cdot \frac{y^2}{4} - y^2
\]

\[
= -\frac{3}{16} y^4 + y^3 - y^2 = g(y)
\]

\[
g'(y) = -\frac{3}{4} y^3 + 3y^2 - 2y
\]

\[
g''(y) = y \left( -\frac{3}{4} y^2 + 3y - 2 \right)
\]

\[
g''(y) = 0:\ \ y = 0 \text{ or }
\]

\[-3y^2 + 12y - 8 = 0
\]

\[3y^2 - 12y + 8 = 0
\]

\[y_{1,2} = \frac{1}{3} \left( 6 \pm \sqrt{36 - 24} \right)
\]

\[= 2 \pm \frac{2\sqrt{3}}{3} \in \text{ none of them is } \in [-2, 2]\]
Problem 6: [5 points] A fly is zooming around a room. Fix one corner of the room, call it the point $O$. Prove that at the moment when the fly is at the maximal distance from $O$, its velocity is perpendicular to the line that connects it to $O$.

Hint: Recall that if the coordinates of a fly at a time $t$ are $(x(t), y(t), z(t))$, then its velocity at the time $t$ is the vector of derivatives $(x'(t), y'(t), z'(t))$.

Let $O$ be the origin. Then $(x(t), y(t), z(t))$ are the coordinates of the fly at the time $t$.

Then the square of its distance from the origin is $f(t) = x^2(t) + y^2(t) + z^2(t)$

At the time to when $f(t)$ is maximal, we must have $f'(t_0) = 0$.

By chain rule,

$$f'(t) = 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)$$

$$= 2 \langle x(t), y(t), z(t) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle$$

So, we get:

$$0 = f'(t_0) = \langle x(t_0), y(t_0), z(t_0) \rangle \cdot \vec{v}$$

Then $\vec{v} \perp \langle x(t_0), y(t_0), z(t_0) \rangle$, which is a vector connecting $O$ to the fly.