Instructor: Julia Gordon, gor@math.ubc.ca

First assignment due Jan 10 (until it's postponed).

Recall: let $O = (0,0,0)$ be the origin. Every point $P = (p_1, p_2, p_3) \in \mathbb{R}^3$ gives rise to the vector $\overrightarrow{OP} = \langle p_1, p_2, p_3 \rangle$. Conversely, every vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$ points from $(0,0,0)$ to $(v_1, v_2, v_3)$.

The **magnitude** of a vector $\mathbf{v}$ is its length. Precisely, if $\mathbf{v} \in \mathbb{R}^2$ (resp. $\mathbf{v} \in \mathbb{R}^3$) then its magnitude is $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}$ (resp. $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$; if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$).

Sometimes $||\mathbf{v}||$

A **unit vector** is a vector of magnitude 1. (In $\mathbb{R}^2$, $\mathbf{v} = \langle v_1, v_2 \rangle$ has magnitude $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}$.)
Example: In $\mathbb{R}^2$, $\langle -1, 0 \rangle$ and $\langle \frac{1}{12}, \frac{1}{12} \rangle$ are unit vectors, but $\vec{v} = \langle 3, -4 \rangle$ is not; it has magnitude $|\vec{v}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$.

However, $\vec{u} = \frac{1}{5} \langle 3, -4 \rangle = \langle \frac{3}{5}, -\frac{4}{5} \rangle$ is a unit vector.

In $\mathbb{R}^3$, some unit vectors are $\langle 0, 0, 1 \rangle$ and $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$, and $\langle \frac{1}{2}, -\frac{1}{4}, \frac{\sqrt{5}}{4} \rangle$.

Generally, if $\vec{v}$ is nonzero (in $\mathbb{R}^2$ or $\mathbb{R}^3$), then $\vec{u} = \frac{1}{|\vec{v}|} \vec{v}$ is a unit vector.

We say that $\vec{u}$ is parallel to $\vec{v}$; meaning there exists a scalar $c \in \mathbb{R}$ such that $c\vec{u} = \vec{v}$.

Whether or not $\vec{v}$ is parallel to anything depends on context.

Example: the vectors $\vec{u} = \langle \frac{2}{5}, -\frac{4}{5}, -1 \rangle$ and $\vec{v} = \langle \frac{1}{4}, \frac{1}{3}, \frac{5}{12} \rangle$ are parallel, because $(-\frac{5}{12})\vec{u} = \vec{v}$.
In \( \mathbb{R}^3 \) we have 3 "standard" unit vectors:
\[ \hat{i} = \langle 1, 0, 0 \rangle, \quad \hat{j} = \langle 0, 1, 0 \rangle, \quad \hat{k} = \langle 0, 0, 1 \rangle \]
(In \( \mathbb{R}^2 \), just have \( \hat{i}, \hat{j} \)).

Every vector in \( \mathbb{R}^3 \) can be resolved into components by writing it as a sum of these vectors:
\[ \langle 2, 4, -5 \rangle = 2\hat{i} + 4\hat{j} - 5\hat{k}. \]

and
\[ \langle 1, 0, -1 \rangle = \hat{i} - \hat{k}. \]

In general:
\[ \langle x, y, z \rangle = x\hat{i} + y\hat{j} + z\hat{k}. \]

Example: A boat travels east (in direction \( \hat{i} = \langle 1, 0 \rangle \)) at 2 m/s, as the current carries it downstream (in direction opposite to \( \hat{j} \)) at \( 2\sqrt{3} \) m/s (approx 3.4 m/s). What is its resulting motion?
Solution: Let $\vec{v} = \langle v_1, v_2 \rangle$ be the boat's velocity vector; it's going to be the sum of the boat's lateral and downstream velocities. The downstream velocity is 
$\langle 2\sqrt{3}, 0 \rangle = \langle 2\sqrt{3}, 0 \rangle$.
Similarly, the lateral velocity is 
$\vec{v}_l = \langle 2, 0 \rangle$.

Hence 
$\vec{v} = \langle 2\sqrt{3}, 0 \rangle + \vec{v}_l = \langle 2\sqrt{3}, 0 \rangle + \langle 2, 0 \rangle = \langle 2, -2\sqrt{3} \rangle$.

What is the magnitude? It is 
$|\vec{v}| = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = 4$

so the boat moves at 4m/s, in direction of 
$\vec{u} = \frac{1}{|\vec{v}|} \vec{v} = \frac{1}{4} \langle 2, -2\sqrt{3} \rangle = \langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \rangle$.

See picture, right:

$N = \arctan \left( \frac{\sqrt{3}/2}{1/2} \right) = 60^\circ$ or $\pi/3$. 
So the boat travels 4m/s, South 30° East.
Dot product.

2 vectors give a scalar.

The dot product of \( \mathbf{u}, \mathbf{v} \) in \( \mathbb{R}^2 \) (resp \( \mathbb{R}^3 \)) is

\[
\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 \quad \text{if} \quad \mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2)
\]

(resp.

\[
\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \quad \text{if} \quad \mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3)
\]

FACTS: (1) \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)

(2) \( |\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u} \)

(3) \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \)

\( \mathbf{u} = (u_1, u_2, u_3) \), \( \mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2 \)

\[
= (\sqrt{u_1^2 + u_2^2 + u_3^2})^2 = |\mathbf{u}|^2
\]
The expression \( \mathbf{u} \cdot \mathbf{w} \) is MEANINGLESS.

**Example:** \( \mathbf{u} + \mathbf{v} = \langle 1, 2, 3 \rangle, \mathbf{w} = \langle 0, 1, -1 \rangle, \mathbf{w} = \langle 7, 3, 3 \rangle \)

Then

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{w} &= 1 \cdot 7 + 2 \cdot 3 + 3 \cdot (-1) = 10 \\
\mathbf{v} \cdot \mathbf{w} &= 0 \cdot 7 + 1 \cdot 3 + (-1) \cdot 3 = 0
\end{align*}
\]

So

\[
(\mathbf{u} \cdot \mathbf{w}) \cdot \mathbf{v} = (10) \mathbf{v} = 10 \mathbf{v} \neq \mathbf{0} \text{, but } \mathbf{u} \cdot (\mathbf{w} \cdot \mathbf{v}) = \mathbf{u} \cdot (0) = \mathbf{0}.
\]

**Observe:**

\[
1|\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})
\]

\[
\begin{align*}
(2) \quad \mathbf{u} \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} \\
(3) \quad &= 1|\mathbf{u}|^2 + |\mathbf{v}|^2 - 2 \mathbf{u} \cdot \mathbf{v}
\end{align*}
\]

On the other hand,

\[
1|\mathbf{u} - \mathbf{v}|^2 = 1|\mathbf{u}|^2 + 1|\mathbf{v}|^2 - 2 \mathbf{u} \cdot \mathbf{v} |\cos \theta|.
\]

Thus:

\[
\mathbf{u} \cdot \mathbf{v} = 1|\mathbf{u}| \cdot |\mathbf{v}| \cos \theta \text{ between them.}
\]

Therefore, we call \( \mathbf{u} \) and \( \mathbf{v} \) **orthogonal** if \( \mathbf{u} \cdot \mathbf{v} = 0 \).