Lines in IR³

- Parametric equation for a line in space.

\[ \begin{align*}
  \vec{v} &= \langle a, b, c \rangle \\
  \overrightarrow{OP}_0 + t \cdot \vec{v} &\rightarrow \text{gives us a vector whose end is on the line.} \\
  \langle x_0+at, y_0+bt, z_0+ct \rangle &\rightarrow P \\
  x(t) &= x_0 + at \\
  y(t) &= y_0 + bt \\
  z(t) &= z_0 + ct
\end{align*} \]

- parametric equation for our line.
Comment: The equation for a plane:

\[ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \]

It is easy to check if a given point lies on the plane or not: plug in the coordinates and see if you get 0.

But if you want a picture of this plane:

(Wolfram alpha)

plot \( 3x+2y+z = 6 \) — gives a picture

How to make it? — computer uses a different way to think about a plane

(Parametric eq. with 2 params.)

For our lines: parametric equation is good for plotting: take different \( t \)'s, plug in, and plot.

But it is hard to check if a given point lies on the line.
Want a way to tell if \((x, y, z)\) lies on the given line?

**Symmetric equation of a line.**

Line is given by a parametric equation:

\[
\begin{align*}
X &= X_0 + at \\
Y &= Y_0 + bt \\
Z &= Z_0 + ct
\end{align*}
\]

Solve for \(t\):

\[
\begin{align*}
t &= \frac{X - X_0}{a} \\
t &= \frac{Y - Y_0}{b} \\
t &= \frac{Z - Z_0}{c}
\end{align*}
\]

Get:

\[
\begin{align*}
\frac{X - X_0}{a} &= \frac{Y - Y_0}{b} &= \frac{Z - Z_0}{c}
\end{align*}
\]

Thus, symmetric equation of our line.
Two comments:

1) Symmetric equation is actually 2 equations:
   (Need 2 linear equations to define a line in IR^3).

\[ \frac{x-x_0}{a} = \frac{y-y_0}{b} \quad \text{vertical plane} \]

\[ \frac{y-y_0}{b} = \frac{z-z_0}{c} \quad \text{plane parallel to x-axis} \]

(When one variable is missing, you get a graph consisting of lines parallel to that axis).

Our line is the intersection of these two planes.

2) What if a, b or c are 0?
   Example: if c = 0 (as in our worksheet):
   \[
   \begin{cases}
   \frac{x-x_0}{a} = \frac{y-y_0}{b} \\
   z = z_0
   \end{cases} \quad \text{2 equations.} \]
Worksheet 2: lines and planes

1. Find an equation of the plane containing the points \( P(3, 0, 0), Q(0, 2, 0) \) and \( R(1,0,0), (0,0,1) \)

   \[ \overrightarrow{PQ} = \langle -3,2,0 \rangle \]
   \[ \overrightarrow{PR} = \langle -3,0,1 \rangle \]

   Normal vector \( \vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & 2 & 0 \\ -3 & 0 & 1 \end{vmatrix} = 2i + 3j + 6k \]

   Answer: \( 2(x-3) + 3y + 6z = 0 \) - using the point \( P \).
   
   Also: \( 2x + 3y + 6(z-1) = 0 \) - simplifies to the same.

2. Find a parametric equation of the line containing the point \( R \) and parallel to the vector \( \overrightarrow{PQ} \).

   Vector form: \( \vec{r}(t) = \overrightarrow{OR} + t \cdot \overrightarrow{PQ} \quad \overrightarrow{PQ} = \langle -3,2,0 \rangle \)
   \[ \vec{r}(t) = \langle 0,0,1 \rangle + t \cdot \langle -3,2,0 \rangle \]

   Coordinates form:
   \[ \begin{cases} x = -3t \\ y = 2t \\ z = 1 \end{cases} \]

   Also:
   \[ \begin{cases} x = 6t \\ y = -4t \\ z = 1 \end{cases} \]

   \[ \frac{x}{6} = \frac{y}{-4} = \frac{z}{1} \]

   \[ \begin{aligned} \text{Symmetric:} \\
   \frac{x}{6} = \frac{y}{-4} = \frac{z}{1} \end{aligned} \]

3. Find an equation for the line of intersection of the planes with equations \( x - y + 2z = 0 \) and \( 3y + z = 0 \).

   **Hint:** how to find a direction vector of this line?

   - It needs to be perpendicular to the normal vectors of both planes.