Today: 1) Implicit differentiation
2) Directional derivatives and the gradient.

1. **Implicit differentiation**

\[ F(x, y, z) = 0 \]

defines \( z \) as an implicit function of \( x, y \)

(also: \( x \) an implicit fun of \( y, z \)).

will work with \( z \) implicit fun of \( x, y \).

Want to find \( \frac{\partial z}{\partial x} \).

Differentiate with respect to \( x \). \(< \) as before.

Use chain rule:

\[ \frac{\partial}{\partial x} F(x, y, z(x, y)) = 0 \]

\[ F_x \cdot \frac{\partial z}{\partial x} + F_y \cdot \frac{\partial y}{\partial x} + F_z \cdot \frac{\partial z}{\partial x} = 0. \]

\[ \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad (y, x \text{ are independent variables}) \]

\[ \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{vertical tangent} \]

\[ F_z \neq 0 \]

\[ x \text{ is a fun of } y, z \]

\[ \frac{\partial z}{\partial x} : \ F_x \cdot \frac{\partial x}{\partial x} + F_y \cdot \frac{\partial y}{\partial x} + F_z \cdot \frac{\partial z}{\partial x} = 0 \]

\[ \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \]

\[ \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \]

All this works if \( F_x \neq 0 \)

\[ x \text{ is not a dir. fun of } y, z \]

\[ F = 0 \leftrightarrow \text{ tangent parallel to } \text{x-axis} \]
Directional derivatives and gradients

Hot air balloon.

\[ T(x, y, z) \] - temperature at \( (x, y, z) \).
\[ (x(t), y(t), z(t)) \] - coordinates at time \( t \) after launch.

Question: What is \( \frac{dT}{dt} \) for you at \( t = t_0 \)?

Given: at \( t = t_0 \), you are at \( (10, 5, 3) \) hundreds of m from the origin.

\[ \frac{dT}{dx} \big|_{(10,5,3)} = 0.2 \text{ °/100 m} \]

\( \text{degrees per 100 m} \)

\[ \frac{dT}{dy} \big|_{(10,5,3)} = -0.3 \text{ °/100 m} \]

\[ \frac{dT}{dz} \big|_{(10,5,3)} = -1 \text{ °/100 m} \]

Your velocity:

\[ \vec{v} = \langle 0.2, 0.1, 0.1 \rangle \text{ in } 100 \text{ m/min} \]

\[ \left. \frac{dT}{dt} \right|_{t=t_0} = \frac{dT}{dx} \left. \frac{dx}{dt} \right|_{t=t_0} + \frac{dT}{dy} \left. \frac{dy}{dt} \right|_{t=t_0} + \frac{dT}{dz} \left. \frac{dz}{dt} \right|_{t=t_0} \]

\( \text{chain rule} \)

The rate of change of temp. you experience
$$P = \left< \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right> \cdot \left< \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right>$$

comes from $T(x,y,z)$

called Gradient vector of $T(x,y,z)$

$$\nabla T = \nabla \cdot \vec{V}$$

$$\nabla T = \left< 0.2, -0.3, -1 \right> \cdot \left< 0.2, 0.1, 0.1 \right> = 0.09$$

our example

in direction of $\vec{u}$

Directional derivative of $T(x,y,z)$ $\vec{V}$ is the rate of change of $T(x,y,z)$ you would experience if you go in the direction of $\vec{u}$ with unit speed.
Worksheet 8: implicit differentiation; directional derivatives

1. Let \( x \) be an implicit function of \( y, z \) defined by the relation
\[
xy^3 + e^{xy + z} = 1 + e^3.
\]
Find \( \frac{\partial x}{\partial z} \) at the point \((1, 1, 2)\).

\[
\begin{align*}
F(x, y, z) &= xy^3 + e^{xy + z} - (1 + e^3) = 0 \\
F_x &= y^3 + ye^{xy + z} \\
F_y &= 3xy^2 + xe^{xy + z} \\
F_z &= e^{xy + z}
\end{align*}
\]

\[
\frac{\partial x}{\partial z} = -\frac{F_z}{F_x} = -\frac{e^{xy + z}}{y^3 + ye^{xy + z}}
\]

To evaluate at \((1, 1, 2)\): plug it in:

\[
\frac{\partial x}{\partial z} \bigg|_{(1,1,2)} = -\frac{e^3}{1 + e^3}
\]

2. Let \( f(x, y, z) = e^{xy} + 3xz \). Let \( v = (1, 2, 3) \). Find \( D_v f \) at the point \((1, 5, 6)\).

\[
D_v f = \nabla f \cdot \hat{u} = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right> \cdot \left< \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right> = \frac{7e^5 + 27}{\sqrt{14}}
\]

\[\hat{u} = \text{unit vector in the direction of } v\]
\[\hat{u} = \frac{v}{|v|} = \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\]

\[\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{7e^5 + 27}{\sqrt{14}}
\]

Evaluate at \((1, 5, 6)\):

\[
\nabla f \bigg|_{(1, 5, 6)} = \left< 5e^5 + 18, e^5, 3 \right>
\]