Recall: Last worksheet told:

\[ f(x, y) = x^3 + xy^2 - y^4 \]

Found critical points: \((0, 0), (-3, \frac{3}{\sqrt{2}}), (-3, -\frac{3}{\sqrt{2}})\)

Want to classify these points.

\[
D = \begin{vmatrix}
    f_{xx} & f_{xy} \\
    f_{xy} & f_{yy}
\end{vmatrix}
\]

\[
f_{xx} = 3x^2 + 2xy^2, \quad f_{xx} = 6x + 2y^2, \quad f_{xy} = 4xy
\]

\[
f_{yy} = 2x^2 - 4y^3, \quad f_{yy} = 2x^2 - 12y^2, \quad f_{yx} \text{ (function is a polynomial so smooth).}
\]

At \((0, 0)\):

\[
D = \begin{vmatrix}
    0 & 0 \\
    0 & 0
\end{vmatrix} \quad \text{undetermined.}
\]

At \((-3, \frac{3}{\sqrt{2}})\):

\[
D = \begin{vmatrix}
    -9 & -\frac{36}{\sqrt{2}} \\
    -\frac{36}{\sqrt{2}} & -36
\end{vmatrix} = 9.36 - \left(\frac{36}{\sqrt{2}}\right)^2 < 0.
\]

\(\text{saddle point}\)

(can also check, \((-3, -\frac{3}{\sqrt{2}})\) is also a saddle point.)
How the classification works

- Key point: Taylor approximation.
  
  Recall: In one variable:
  
  \[ f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \]
  
  For \( x \) close to \( a \) ↑ linear approximation
  
  quadratic term (tells you about concavity).

For functions of two variables:

\[ f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2!}(x-a)^2 + \frac{f_{yy}(a,b)}{2!}(y-b)^2 + \frac{f_{xy}(a,b)}{2!}(x-a)(y-b) \]

↑ near \((a,b)\) Linear approximation

quadratic term

= \((x-a)^2 + (y-b)^2 + (x-a)(y-b)\)

come from \( f_{xx}, f_{xy}, f_{yy} \)
For \( f(x,y) \): the graph of this is the paraboloid that best approximates graph of \( f(x,y) \). 

\[ f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \text{quadratic term} \]

linear approximation \( o \) at a critical point.

**quadratic term** = \( (x-a)^2 + (y-b)^2 + (x-a)(y-b) \)

made from \( f_{xx}, f_{yy}, f_{xy} \).

What kind of paraboloid do we get?

\[ z = Ax^2 + By^2 + Cxy \]

**Fact:** \( D = \begin{vmatrix} A & C \\ C & B \end{vmatrix} > 0 \) \( \Rightarrow \) elliptic \( \checkmark \)

\[ z = x^2 - y^2 \]

\( D < 0 \) - hyperbolic paraboloid.

critical pt: local max in one direction
local min in another
**Upshot:** $D$ tells you the shape of the paraboloid that best approximates the graph near $(a, b)$.

$D > 0$:
- $f_{xx} > 0$ (local minimum)
- $f_{xx} < 0$ (local maximum)

$D < 0$:
- Saddle point

$D = 0$:
- Plane is the best approx to your graph.

**Example:**

$f(x, y) = x^2$

- $f_x = 2x$, $f_y = 0$

All points $x = 0$ (the whole $y$-axis) are critical.

$D = 0$ at all these points.
How to recognize critical points on contour plots:

Local max or min (can tell which only if the contours are labelled)

Hyperbolas

"Prototype saddle"

Zoom in

The two touching circles are a single level curve
Worksheet II: Absolute max/min

Produce the complete list of points where the absolute max or min of \( f(x, y) = x^2 + 3xy - 2y^2 \) on the triangle \( T \) with vertices \((-1, 2), (1, -1) \) and \((2, -1)\) could occur; do not evaluate the function at these points.

- **Step 1:** Find critical points,
  \[
  \begin{align*}
  f_x &= 2x + 3y \\
  f_y &= 3x - 4y
  \end{align*}
  \]

  \[
  \begin{cases}
  2x + 3y = 0 \\
  3x - 4y = 0
  \end{cases} \\text{The only solution is:} \ (0, 0) \]

- **Step 2:** Analyze the boundary.
  - Piece ①: \( x = -1 \), \(-1 \leq y \leq 2\)
  - Piece ②: \( y = -1 \), \(-1 \leq x \leq 2\)
  - Piece ③:

- **Step 3:** Do not forget to include the vertices.
Piece 1

\[ x = -(\_\_), \quad -1 \leq y \leq 2 \]

Plug it in:

\[ f(-1, y) = (-1)^2 + 3(-1)y - 2y^2 \]

\[ g(y) = 1 - 3y - 2y^2 \quad \text{← function of } y. \]

We have to look for its critical points in \([-1, 2]\).

\[ g'(y) = -3 - 4y \quad \text{and } g'(y) = 0; \quad y = -\frac{3}{4}. \]

\[ \text{Calculus: } \quad \begin{cases} \text{Get: } & \left[-1, -\frac{3}{4}\right] \end{cases} \quad \text{← goes on the list.} \]

Piece 2

Plug in \( y = -1, \quad -1 \leq x \leq 2 \).

Get:

\[ g_2(x) = f(x, -1) = x^2 + 3x \cdot (-1) - 2 (-1)^2 = x^2 + 3x - 2. \]

\[ g_2'(x) = 0; \quad 2x + 3 = 0, \quad x = -\frac{3}{2}. \]

\[ \text{Get: } \quad \left[ -\frac{3}{2}, -1 \right] \]

Piece 3: Need the equation of the line:

\[ y = -x + 1 \]

Get:

\[ g_3(x) = f(x, -x + 1) \]

\[ = x^2 + 3x(-x + 1) - 2(-x + 1)^2 \]

\[ = -4x^2 + 7x - 2 \]

After simplifying,

\[ g_3'(x) = -8x + 7, \quad x = \frac{7}{8}. \]

Then \( y = -\frac{7}{8} + 1 = \frac{1}{8}. \)

\[ \text{Get: } \quad \left[ \frac{7}{8}, \frac{1}{8} \right] \]

Final answer: The list of points where absolute max or min could be is:

\[ (90); \quad \text{critical point inside } T \]

\[ (-1, -\frac{3}{4}), \left(\frac{3}{2}, -1\right), \left(\frac{7}{8}, \frac{1}{8}\right); \quad \text{from the edges of } T \]

\[ (-1, 0), \left(1, 0\right); \quad \text{vertices of } T \]