Some additional points about double integrals

- If you have \( \int_a^b \int_c^d f(x) g(y) \, dx \, dy \)

\( \begin{array}{c}
s = \text{functions of a single variable} \\
\text{limits} \\
\text{are} \\
\text{all constants}
\end{array} \)

\( = \left( \int_a^b g(y) \, dy \right) \left( \int_c^d f(x) \, dx \right) \)

This also works for \( \int \int f(r) g(\theta) \, dr \, d\theta \)

if the limits are constant.

- When switching the order of integration, check:
  - Is the function positive on the domain?
  - If no, is its absolute value bounded?
  - If no to all, be careful! Put \( |f| \) first, check if integral is finite.
2. Classic example: the 'total mass' of the Gaussian distribution. Find

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx. \]

**Hint:** we write:

\[ \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 = \int_{-\infty}^{\infty} e^{-x^2} \, dx \int_{-\infty}^{\infty} e^{-y^2} \, dy = \iint_{\mathbb{R}^2} e^{-x^2-y^2} \, dx \, dy. \]

Now change to polar coordinates.

Think about \( \int \) as \( \lim_{R \to \infty} \int_{R^2} \).

For this to work, thinking of \( \int \int_{R^2} e^{-x^2-y^2} \, dx \, dy \) as \( \lim_{A \to \infty} \int \int_{A} e^{-x^2-y^2} \, dx \, dy \).

(Can prove that both are true and give the same number.)

\[ = \lim_{R \to \infty} \int \int_{\text{disk of radius } R} e^{-x^2-y^2} \, dx \, dy = \lim_{R \to \infty} \int_{0}^{2\pi} \int_{0}^{R} e^{-r^2} r \, dr \, d\theta \]

\[ = \lim_{R \to \infty} \int_{0}^{2\pi} \frac{1}{2} \, d\theta \]

\[ = \lim_{R \to \infty} \int_{0}^{2\pi} \frac{1}{2} \, d\theta \]

\[ = \frac{1}{2} \int_{0}^{2\pi} 1 \, d\theta \]

\[ = \frac{1}{2} \cdot 2\pi \]

\[ = \pi \]

\[ \int_{0}^{2\pi} \int_{0}^{R} e^{-r^2} r \, dr \, d\theta \]

\[ = \frac{1}{2} \int_{0}^{2\pi} \left[ -e^{-r^2} \right]_{0}^{R} \, d\theta \]

\[ = \frac{1}{2} \int_{0}^{2\pi} \left( 1 - e^{-R^2} \right) \, d\theta \]

\[ = \frac{1}{2} \cdot 2\pi \left( 1 - e^{-R^2} \right) \]

\[ = \pi \left( 1 - e^{-R^2} \right) \]

Works when the substitution is an increasing & decreasing function on the interval \( 0 \leq r \leq R \).

\( 2^2 \) means \( 0^2 \leq u \leq R^2 \).
\[
\lim_{n \to \infty} \frac{2}{\pi} \int_0^{\infty} e^{-u^2} du = \pi \cdot \ln(e^{-u} \bigg|_0^{\infty}) = \pi \cdot (1 - 0) = \pi
\]

Answer: \[\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\]

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**Triple integrals**

A function \( f(x, y, z) \); want to integrate it over a solid in \( \mathbb{R}^3 \).

\[
\iiint_E f(x, y, z) \, dV = \lim_{P \to \text{volume}} \left( \sum f(x_i^*, y_i^*, z_i^*) \cdot V_{\text{cube}} \right)
\]

Sum over little cubes
Example  Find the average value of 
\[ f(x,y,z) = \frac{1}{8} (x+y-z) \]
over the tetrahedron bounded by the 
coordinate planes and the plane \( 3x+2y+z = 6 \).

To sketch: find its intercepts with the axes.
(set: \( y = z = 0 \))
(solve for \( x \):
\[ 3x = 6 \quad x = 2 \]
Average: in 1 dim: average of \( f(x) \) over \([a,b]\)

\[
3 = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

in 2 dim: average of \( f(x,y) \) over \( D \) is

\[
\frac{1}{\text{area}(D)} \iint_D f(x,y) \, dA
\]

in 3 dim: average of \( f(x,y,z) \) over \( E \) is

\[
\frac{1}{\text{Volume}(E)} \iiint_E f(x,y,z) \, dV.
\]

So we need to find:

1) Volume of our tetrahedron

2) \( \iiint_E (x+y+z) \, dV \)

Everywhere about \( E \) will be encoded in the limits of integration.
1) Volume of $E$:

2 ways:

1) think of $E$ as lying under the graph of $z = 6 - 2y - 3x$ (blue plane) over the bottom face.

Equation: set $z = 0$:

$$3x + 2y = 6$$

$$y = 3 - \frac{3}{2}x$$

2) or: $V = \iiint_E 1 \, dV$
Let us set it up as \( \iiint_E 1 \, dV \)

\[
= \iiint_2 \left( \int_{z=0}^{z=6-3x-2y} 1 \, dz \right) \, dA \quad \text{in xy-plane}
\]

(\text{this agrees with the first approach:} \quad V = \iint (6-2y-3x) \, dA)

\[
= \int_{x=0}^{x=2} \int_{y=0}^{y=3-1/2x} \int_{z=0}^{z=6-3x-2y} 1 \, dz \, dy \, dx
\]

\[
= \int_0^2 \int_0^{3-1/2x} \int_0^{6-3x-2y} 1 \, dz \, dy \, dx
\]

\[
= \int_0^2 \int_0^{3-1/2x} (6-2y-3x) \, dy \, dx
\]

\[
= \ldots \text{compute as usual}
\]