Today: **Quadric surfaces. (and cylinders)**

**Cylinders**: a surface consisting of parallel lines.

- we are used to: ![cylinder](image)
- doesn't have to be round!
  - (doesn't have to be "straight")

1. pick any direction
2. draw a line parallel to your chosen direction through every point of the curve.

The surface you get is called a **cylinder** (on \(C\)).

**Example**: our usual cylinder is obtained from a circle (circular)

![cylinder](image)

\(\leftarrow\) **parabolic cylinder.**
Quadratic surfaces: surfaces defined by a quadratic equation in \( x, y, z \).

\[
a x^2 + b y^2 + c z^2 + d x \cdot y + e x \cdot z + f y z + g \cdot x + h \cdot y + l \cdot z + m + n + o p = 0
\]

(a, b, c, d, e, ..., p are coefficients (constants))

General form of an equation of such a surface.

Example: 1) \( x^2 + y^2 + z^2 = R^2 \) - sphere of radius \( R \) centered at \( 0 \).

2) \( x^2 + 2x + y^2 + z^2 = R^2 \)

\[
(\text{x+1})^2 - 1 + y^2 + z^2 = R^2
\]

\[
(\text{x+1})^2 + y^2 + z^2 = R^2 + 1 \quad \text{- sphere centered at } (-1, 0, 0) \text{ of radius } \sqrt{R^2 + 1}
\]

Square of the distance from \((x, y, z)\) to \((-1, 0, 0)\)
Example 3) Recall webwork: set of points equidistant from $P_1$ and $P_2$ starts out as a quadric:

\[
\begin{align*}
(x-a)^2 + (y-b)^2 + (z-c)^2 &= (x-d)^2 + (y-e)^2 + (z-f)^2
\end{align*}
\]

Then all square terms cancel, get a linear equation defining a plane.

4) What if one of the variables is not participating:

- $x^2 + y^2 = 1 \quad \text{circular cylinder parallel to the } z\text{-axis}$
- $x^2 - y = 0 \quad \text{parabolic cylinder parallel to } z\text{-axis}$ (picture on p. 1)
- $z^2 + x = 0 \quad \text{parabolic cylinder parallel to } y\text{-axis}$
Standard examples:

1. \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]  
   \[ \text{ellipsoid} \]

2. \[ \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]  
   \[ (\text{when } a=b=c=1: \; z=x^2+y^2) \]  
   \[ (\text{elliptic}) \text{ paraboloid}. \]  
   \[ \text{horizontal traces are ellipses}. \]

3. \[ \frac{z^2}{c^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \]  
   \[ \text{hyperbolic paraboloid} \]

4. \[ \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]  
   \[ \text{cone} \]

5. \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]  
   \[ \text{hyperboloid of one sheet} \]

6. \[ -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]  
   \[ \text{hyperboloid of two sheets} \]
   \[ \text{(Read! in 10.1)} \]

Friday

Quiz: lines, planes, maybe quadric surfaces.

Fact: Any general equation of a quadric surface can be reduced to one of these, or an equation defining a plane or a pair of planes.

(see the next two pages for the explanation of 2 and 3)
For (2): Consider slices with horizontal planes:

Let $z = k$, plug it in:

$$k = x^2 + y^2$$

So the trace of our surface on this plane is a circle of radius $\sqrt{k}$.
When $k = 0$, just get $(0, 0, 0)$.

For (3): Consider traces on horizontal planes:

$t = k$
(take $a = b = c = 1$)

Get:

$$k = x^2 - y^2$$

$$k = (x-y)(x+y)$$

Take: $u = x-y$, $v = x+y$

So: horizontal traces are hyperbolas.

Now, consider vertical planes: Fix $X = k$:

Get: $z = k^2 - y^2$ - down-facing parabola

Fix $y = k$: $z = x^2 - k^2$ - upward-facing parabola

See next page for a picture.
The point: later we can consider linear (tangent plane) and quadratic approximations of functions of two variables.

Their graphs will be surfaces like this:

\[ z = ax^2 + by^2 \] depending on whether \( a, b \) are the same sign or not, you get:

\[ \cap \] or \[ \cup \] or \[ \bowtie \] ← saddle