Spherical coordinates

Recall:

\[ x = \rho \sin \varphi \cos \theta \]
\[ y = \rho \sin \varphi \sin \theta \]
\[ z = \rho \cos \varphi \]

\[ r = \sqrt{x^2 + y^2} = \rho \sin \varphi \]

What about volume?

\[ dV = \rho^2 \sin \varphi \ d\varphi \ d\psi \ d\theta \]

in any order

Put it all together:

Volume of our piece

\[ V \approx \Delta \varphi \cdot \rho \sin \varphi \cdot \Delta \theta \cdot \Delta \psi \]

\[ = \rho^2 \sin \varphi \ \Delta \varphi \ \Delta \theta \ \Delta \psi \]

Why the volume factor?

"spherical wedges":

- arc of a circle of radius \( \rho \) correspond to the angle \( \Delta \varphi \)

- its length \( \approx \rho \Delta \varphi \)

horizontal circles

angular measure \( \Delta \theta \)

radius of these circles \( \approx \rho \sin \varphi \)

length: \( \approx \rho \sin \varphi \ \Delta \theta \)
\( y = \text{const} : \text{cone} \)
\( \rho = \text{const} : \text{sphere} \)
\( \Theta = \text{const} : \text{half of a vertical plane} \)

Example: Use spherical coordinates to find the \( z \)-coordinate at the centroid of a cone with angle 30° at the vertex, of radius 2 at the base.

\[
\bar{z} = \frac{1}{\text{Vol (cone)}} \iiint_{\text{cone}} z \, dV
\]
\[ \text{vol}(\text{cone}) = \iiint_{\text{cone}} 1 \, dv = \iiint_{\text{cone}} \rho^2 \sin \phi \, dp \, d\rho \, d\theta \]

Limits: \( 0 \leq \theta \leq 2\pi \) - cross-sections by horizontal planes are circles centred at \( 0 \).

\( 0 \leq \phi \leq 30^\circ \) (have to use radians!)

\[ \frac{\pi}{6} \]

- this gives our cone

Wrong thing to try:

\[ \iiint_{\text{cone}} \rho^\frac{1}{2} \, dp \, d\rho \, d\theta \]

Need to find \( d \):

\[ d \sin 30^\circ = 2 \]

\[ d = 4 \]

How to correct?
Need the equation of this horizontal plane in spherical coördinates.

The plane is given by

\[ z = 2 \sqrt{3} \]

\[ z = s \cos \psi \]

So:

\[ s \cos \psi = 2 \sqrt{3} \]

Correct integral:

\[
\int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_0^{2\sqrt{3}/\cos \psi} \frac{1}{r^2} r^2 \sin \phi \, dr \, d\psi \, d\theta = V_{\text{orl}}.
\]

\[ V_{\text{ol}} = \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_0^{2\sqrt{3}/\cos \psi} r^2 \sin \phi \, dr \, d\psi \, d\theta. \]

\[ \bar{z} = \left( \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_0^{2\sqrt{3}/\cos \psi} r^2 \cos \psi \cdot r^2 \sin \phi \, dr \, d\psi \, d\theta \right) \cdot \frac{1}{V}
\]

Note: here total mass = volume (so we are using \( \frac{1}{V} \) for centre of mass because density = 1.)